NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS | WWW.NCTM.ORG

MATHEMATICS teacher

NOVEMBER 2012

FOCUS ISSUE

fostering flexible mathematical thinking

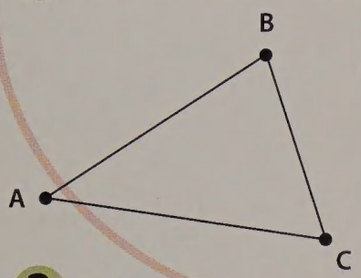
Marygrove College Library
8425 West McNichols Road
Detroit, MI 48221

Tired of using the wrong drawing tool for your math documents?

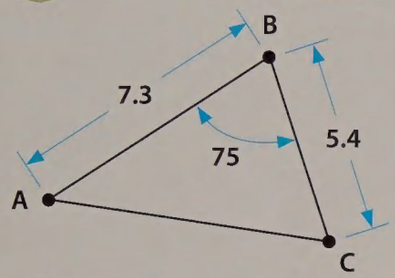
Now you can create high quality math diagrams in a fraction of the time!
Works seamlessly with Word® and PowerPoint® on PC or Mac.

- Natural, intuitive way to create math drawings.
- Quickly create accurate, to-scale figures.
- Easily paste your finished figures into tests and worksheets or use them in PowerPoint® presentations.
- With a single click, add leader lines and dimension arrows to specify lengths, angles, distances and radii.
- Requires PC running Windows® XP, Vista®, Windows® 7, or Intel based Macintosh®.

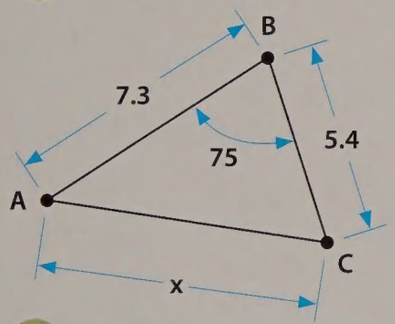
1 Sketch the figure



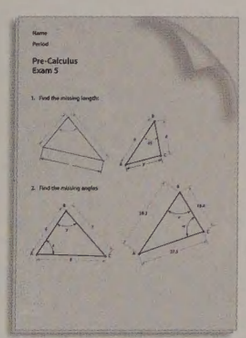
2 Add constraints



3 Create annotations

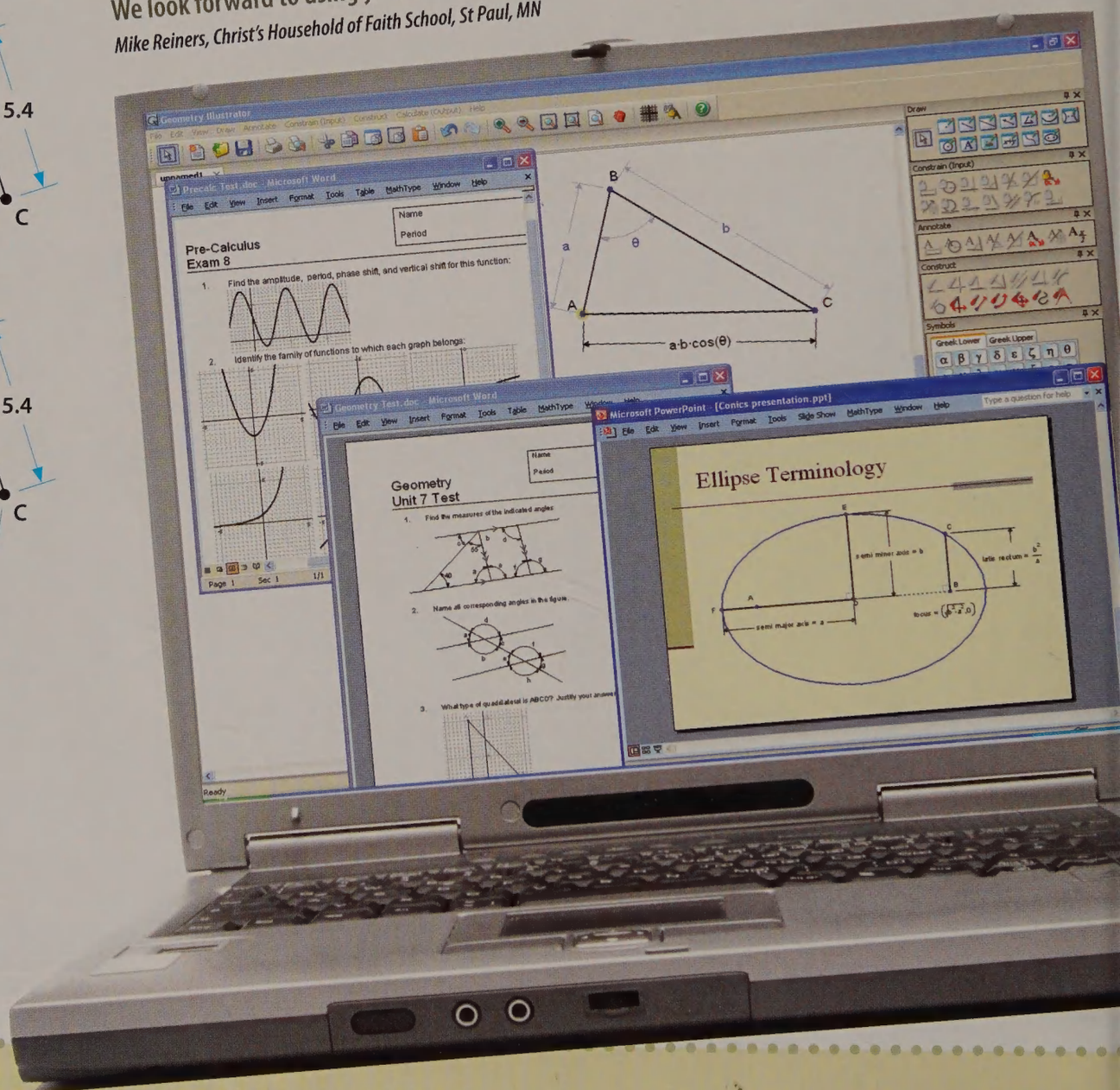


4 Copy and paste



"We gave Math Illustrations to the Minnesota State High School Mathematics League problem writing team to use on problems for the regular season and the state tournament. The program also worked very well for drawing the figures used during the Math Bowl at the tournament. We look forward to using your software for many years to come."

Mike Reiners, Christ's Household of Faith School, St Paul, MN



MATH ILLUSTRATIONS

For more information and to place an order, go to
www.mathillustrations.com

Saltire Software

contents

Volume 106, Number 4, November 2012

features

260 **Using Multiple Representations to Teach Composition of Functions**
Scott Steketee and Daniel Scher
Experience with multiple representations fosters students' robust understanding of what functions are, how they behave, and how they can be composed.

269 **Thinking Like a Mathematician**
Michael K. Weiss and Deborah Moore-Russo
The moves that mathematicians use to generate new questions can also be used by teachers and students to tie content together and spur exploration.

274 **Understanding the Derivative through the Calculus Triangle**
Eric Weber, Michael Tallman, Cameron Byerley, and Patrick W. Thompson
We know derivatives are about rates. Why don't our students?

288 **Exploring Quadrilaterals through Flexible Approaches**
Kerri Richardson, Anne Reynolds, and Catherine S. Schwartz
Rich mathematical tasks—here, finding and categorizing the quadrilaterals that can be made with vertices on a 4×4 grid—can promote adaptability in any classroom.

295 **Looking for Connections between Linear and Exponential Functions**
Jane-Jane Lo and James L. Kratky
Students compare and contrast linear and exponential relationships from tabular, graphical, and algebraic representations.

302 **Tasks That Make Connections through Representations**
Joe Garofalo and Christine P. Trinter
Students think resiliently about using the quadratic formula, analyzing factors graphically, finding the shortest distance between two points, and finding margin of error.

on the cover

The call for manuscripts for the 2012 Focus Issue was designed to attract submissions that help promote both NCTM's Process Standards and the Standards for Mathematical Practice of the Common Core State Standards. The six articles here have achieved that goal admirably well. Working between and among multiple representations, looking for alternate routes to a problem solution when faced with obstacles, and posing and solving problems are front and center in this issue.

COVER PHOTO: SPIKE MAFFORD/GETTY IMAGES

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

ALEX PACHA/ISTOCKPHOTO

MATHEMATICS teacher

EDITORIAL PANEL

SARAH SCHUHL, Centennial High School, Gresham, Oregon; *Chair*
DANE R. CAMP, 'Iolani School, Honolulu, Hawaii; *Board of Directors Liaison*
DAN CANADA, Eastern Washington University, Cheney
DAVID CUSTER, Decatur City High School, Decatur, Georgia
GREGORY D. FOLEY, Ohio University, Athens, Ohio
ALLISON GOLDEN, Champlain Valley Union High School, Hinesburg, Vermont
STAN IZEN, Latin School of Chicago, Chicago (*retired*)
SANDRA R. MADDEN, University of Massachusetts–Amherst
LAURIE H. RUBEL, Brooklyn College, CUNY, New York
GREG STEPHENS, Hastings High School, Hastings-on-Hudson, New York
ALBERT GOETZ, NCTM; *Staff Liaison*

JOURNALS STAFF

KEN KREHBIEL, *Associate Executive Director for Communications*
JOANNE HODGES, *Director of Publications*
ALBERT GOETZ, *Journal Editor*
GRETCHEN SMITH MUI, *Editor*
LUANNE M. FLOM, PAMELA A. GRAINGER, ELIZABETH M. SKIPPER, SARA-LYNN GOPALKRISHNA, *Contributing Editors*
SHEILA J. BARKER, *Review Services Assistant*
CHRISTINE A. NODDIN, *Publications Assistant*

MARKETING STAFF

JENNIFER J. JOHNSON, *Senior Director, Member Services, Marketing, and Business Development*
TOM PEARSON, *Sales Manager*

NCTM BOARD OF DIRECTORS

LINDA M. GOJAK, John Carroll University, University Heights, Ohio; *President*
J. MICHAEL SHAUGHNESSY, Portland State University, Oregon; *Past President*
KICHOON YANG, NCTM; *Executive Director*
ROBERT Q. BERRY III, University of Virginia, Charlottesville
MARGARET (PEG) CAGLE, Lawrence Gifted Magnet School, Los Angeles Unified School District
DANE R. CAMP, New Trier High School, Winnetka, Illinois
ANNE M. COLLINS, Lesley University, Cambridge, Massachusetts
DEBBIE DUVALL, Elk Island Public Schools, Sherwood Park, Alberta
MARK W. ELLIS, California State University, Fullerton
KAREN J. GRAHAM, University of New Hampshire, Durham
GLADIS KERSAINT, University of South Florida, Tampa
LATRENDA KNIGHTEN, Polk Elementary School, Baton Rouge, Louisiana
MATTHEW R. LARSON, Lincoln Public Schools, Nebraska
JONATHAN (JON) WRAY, Howard County Public Schools, Maryland

To contact a journal staff person, e-mail **mt@nctm.org**.

Mission Statement: The National Council of Teachers of Mathematics is a public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

All correspondence should be addressed to the *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Manuscripts should be prepared according to the *Chicago Manual of Style* and the United States Metric Association's *Guide to the Use of the Metric System*. No author identification should appear on the manuscript; the journal uses a blind-review process. To send submissions, access **mt.msubmit.net**. Send letters to the editor to **mt@nctm.org**.

Permission to photocopy material from *Mathematics Teacher* is granted to persons who wish to distribute items individually (not in combination with other articles or works), for educational purposes, in limited quantities, and free of charge or at cost; to librarians who wish to place a limited number of copies on reserve; to authors of scholarly papers; and to any party wishing to make one copy for personal use. Permission must be obtained to use journal material for course packets, commercial works, advertising, or professional development purposes. Uses of journal material beyond those outlined above may violate U.S. copyright law and must be brought to the attention of the National Council of Teachers of Mathematics. For a complete statement of NCTM's copyright policy, see the NCTM website, **www.nctm.org**.

For information on **article photocopies or back issues**, contact the Customer Service department in the Headquarters Office.

A cumulative index appears on the NCTM website at **www.nctm.org/mt/mt-indexes.htm**. The *Mathematics Teacher* is indexed in *Biography Index, Contents Pages in Education, Current Index to Journals in Education, Education Index, Exceptional Child Education Resources, Literature Analysis of Microcomputer Publications, Mathematical Reviews, Media Review Digest,* and *Zentralblat für Didaktik der Mathermatik*.

Information is available from the Headquarters Office or online at www.nctm.org/membership regarding the three **other official journals**, *Teaching Children Mathematics*, *Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of NCTM's services. Dues for individual membership are $81 (U.S.), which includes $34 for each *Mathematics Teacher* subscription. Each additional school journal (*Mathematics Teaching in the Middle School* and *Teaching Children Mathematics*) subscription is $34. Each additional subscription to the *Journal for Research in Mathematics Education* is $61. Foreign subscribers add $18 (U.S.) postage for the first journal and $4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

The *Mathematics Teacher* (ISSN 0025-5769) (IPM 1213431) is published monthly except June and July, with a combined December/January issue, by the National Council of Teachers of Mathematics at 1906 Association Drive, Reston, VA 20191-1502. Periodicals postage paid at Herndon, Virginia, and at additional mailing addresses. POSTMASTER: Send address changes to *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Telephone: (703) 620-9840; orders: (800) 235-7566; fax: (703) 476-2970; e-mail: nctm@nctm.org. © 2012. The National Council of Teachers of Mathematics, Inc. Printed in the U.S.A.

departments

245 **From the Editorial Panel**
Fostering Flexible Mathematical Thinking

246 **Reader Reflections**

250 **Media Clips**
The Burglar in the Suitcase //
Climate History in an Ice Core
Kristyn K. Wilson // Chris Achong
Edited by Louis Lim and Lionel Garrison

254 **Mathematical Lens**
A Road for Every Wheel
Douglas Wilcock
Edited by Ron Lancaster and Brigitte Bentele

280 **December 2012 Calendar and Solutions**
Edited by Margaret Coffey and Art Kalish

308 **Connecting Research to Teaching**
Keys to Successful Group Work:
Culture, Structure, Nurture
Kasi C. Allen
Edited by Margaret Kinzel and Laurie Cavey

313 **Delving Deeper**
Close Encounters with Stirling Numbers
of the Second Kind
Martin Griffiths
Edited by Maurice Burke, J. Kevin Colligan, Maria Fung, and Jeffrey J. Wanko

318

LOKI BAHO/ISTOCKPHOTO

318 **For Your Information**
Publications

320 **The Back Page: My Favorite Lesson**
Sliding through Logarithms
William J. Wolbert
Edited by Jennifer Wexler

calls for manuscripts

244 **2014 Focus Issue: What Makes Ideas Stick?**

299 **Technology Tips**

315 **Connecting Research to Teaching**

more4U

Look online for these additional items:

- Lo and Kratky (p. 295): Interactive GeoGebra applets
- Mathematical Lens (p. 254): An additional photograph for your own mathematical questions

Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mt.**

in the next issue

Coming in December 2012/January 2013:

- "December 21, 2012: A Date of Opportunity," by Shirley B. Gray and Zebanya Rice
- "Authentic Tasks in a Standards-Based World," by Michael Todd Edwards, Suzanne R. Harper, and Dana C. Cox
- "Rate Problems: Thinking across the Curriculum," by Patrick Kimani and Nicole Engelke

CALL FOR MANUSCRIPTS

2014 Focus Issue

What Makes Ideas Stick?

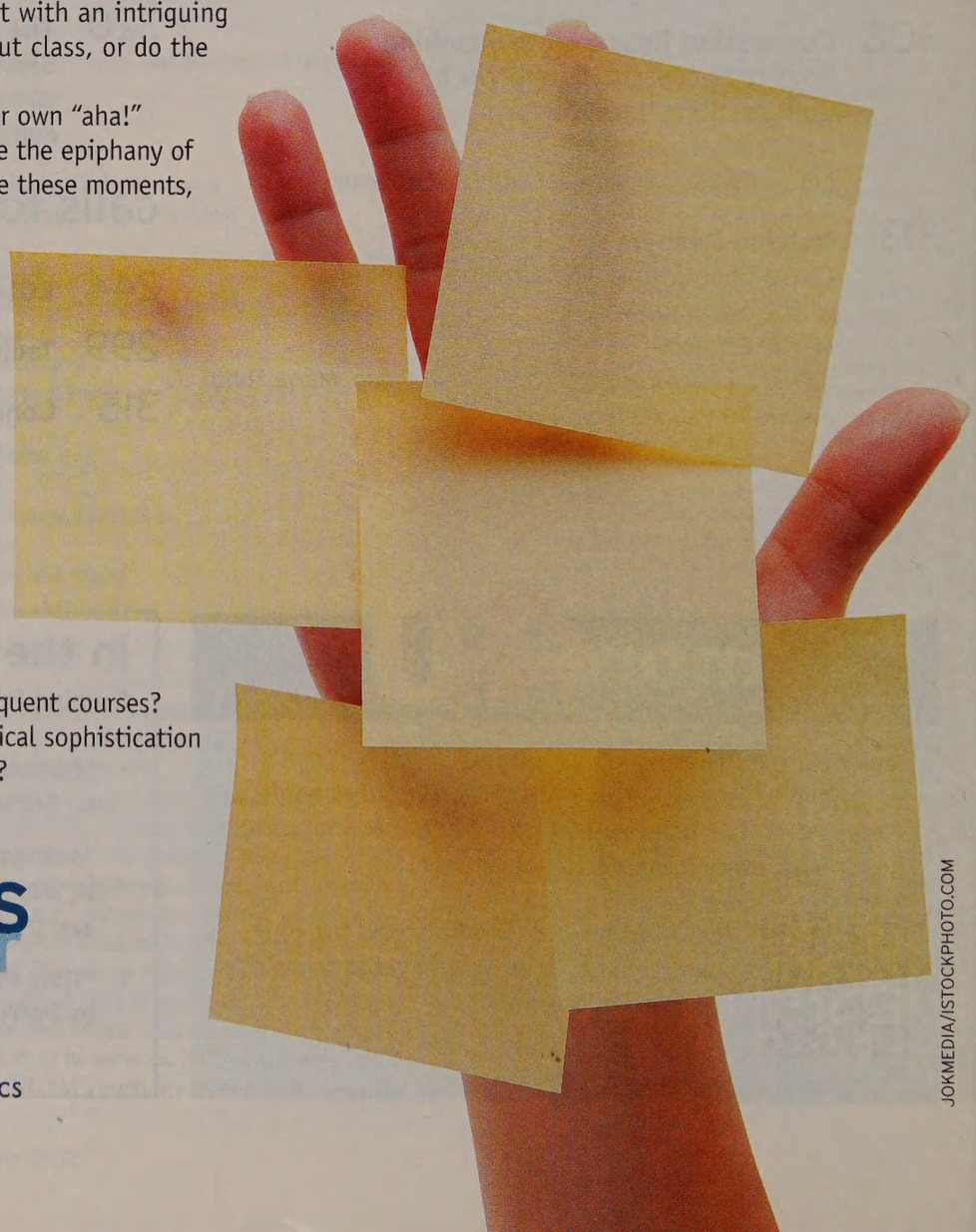
Teachers design lessons to engage students with big ideas of mathematics. Does engagement necessarily imply that students will retain what we hope they will retain? How do students in your classes develop an appreciation for mathematical big ideas and the relationships among them? What lesson characteristics seem to promote the permanence of concepts in the minds of your students?

What works? Write about your big ideas—*big* big ideas or *small* big ideas—that help make mathematics stick. Following are a few suggestive, but not prescriptive, prompts:

- Great lessons stay in the minds of students. What lessons do you have that engage your students in exciting and meaningful mathematical experiences?
- Some ideas reside with students because of the instruction that introduces them more than the inherent mathematics. Do you start with an intriguing idea and then develop it throughout class, or do the big ideas build in the background?
- Students often surprise us with their own "aha!" moments, usually as they experience the epiphany of an unexpected connection. What are these moments, and how do we foster these connections and epiphanies?
- How do reasoning and sense making evolve to foster mathematical memory? What are high-leverage practices that help ideas stick?
- What kinds of challenges do big mathematical ideas face in the current political climate?
- How can technology impact the "stickiness" of ideas?
- How do we ensure that students internalize the mathematics that they need to be successful in subsequent courses? How do they develop the mathematical sophistication to adapt to an ever-changing world?

Contribute to this special *MT* issue by highlighting mathematical and pedagogical big ideas from your own teaching. Show us what helps students retain ideas and what leads them to an appreciation of broad concepts.

The *MT* Editorial Panel invites submissions for the 2014 Focus Issue. Please submit manuscripts at **mt.msubmit.net** by **May 1, 2013**. Be sure to check the box indicating that this manuscript is for the 2014 Focus Issue entitled What Makes Ideas Stick? Guidelines for the preparation of manuscripts can be found at www.nctm.org/publications/content.aspx?id=22602. No author identification should appear in the manuscript's article file. If you have ideas related to this topic and would like to discuss them before sending a manuscript, please contact Albert Goetz, agoetz@nctm.org.



JOKMEDIA/ISTOCKPHOTO.COM

MATHEMATICS teacher

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

fostering flexible mathematical thinking

Understanding and making connections is at the heart of flexible mathematical thinking. Flexible thinking is generally observed in students' ability and willingness to work with multiple representations, but it can also be seen in their facility in adapting problem-solving strategies when faced with novel situations, reversing thought processes, viewing notation as a process as well as an object, interpreting others' ideas, and posing problems. Although teachers may characterize flexible mathematical thinking differently according to context, we can probably all agree that providing students with opportunities to engage in flexible mathematical thinking is vital for fostering the kind of mathematical understanding that we want them to have.

The 2012 Focus Issue of *Mathematics Teacher* centers on classroom activities and purposeful tasks that support flexible mathematical thinking. These tasks and activities can promote NCTM's Process Standards and the Standards for Mathematical Practice set forth in the Common Core State Standards. These practices include making sense of problems, reasoning abstractly and quantitatively, constructing viable arguments and critiquing the reasoning of others, modeling with mathematics, using appropriate tools strategically, attending to precision, looking for and making structure, and looking for and expressing regularity in repeated reasoning.

One theme that emerges in this issue is how students use and connect multiple representations of a concept. In "Using Multiple Representations to Teach Composition of Functions," Scott Steketee and Daniel Scher detail how students explore the composition of a dilation and a translation through functional, dynagraph, numeric, symbolic, and Cartesian representations. And in "Tasks That Make Connections through Representations," Joe Garofalo and Christine P. Trinter share four tasks in which students can incorporate different representations and thought processes.

The role of persistence—a second theme—is highlighted in "Exploring Quadrilaterals through Flexible Approaches." Here Kerri Richardson, Anne Reynolds, and Catherine S. Schwartz illustrate how in-service teachers determined how many four-sided figures could be constructed on a 4×4 grid. The teachers' work included abandoning an idea and then reconsidering it. In addition, Jane-Jane Lo and James L. Kratky explore connections between functions as a means to a better understanding of rate of change in "Looking for Connections between Linear and Exponential Functions." One especially interesting aspect of this article is how students' facility with graphical, numerical, and algebraic representations helps them become more flexible with the concept of rate of change.

The concept of rate of change is also the subject of "Understanding the Derivative through the Calculus Triangle" by Eric Weber, Michael Tallman, Cameron Byerley, and Patrick W. Thompson. The authors explain how practicing teachers incorporated the sliding calculus triangle as an interactive tool to help students come to a deeper understanding of how the derivative represents the rate of change of a function.

The fourth theme of this issue is that of habits of mind. "Thinking Like a Mathematician," by Michael K. Weiss and Deborah Moore-Russo, describes how problem posing in conjunction with problem solving allows students to engage in the type of flexible thinking required by a mathematician. Examples are provided to illustrate how creating new problems from old can be an effective classroom strategy.

We hope that you will integrate some of these ideas into your teaching. Your students' mathematical thinking will become more flexible, and their mathematical understanding will be richer.

RUDY'S TRIANGLE AND OTHER PASCAL-LIKE TRIANGLES

In "Boundaries in Higher Dimensions and Their Kinship with Combinatorics" (Delving Deeper, *MT* Dec. 2011/ Jan. 2012, vol. 105, no. 5, pp. 391–95), Kate Nowak does an excellent job of depicting the joy and excitement one experiences when investigating mathematics not often found in the usual classroom or textbook. Certainly hyperspace and Pascal's triangle are fertile fields for these endeavors.

A few years ago, a colleague and I were also privileged to discover Rudy's triangle (see "Row and Rising Diagonal Sums for a Type of Pascal Triangle," *Fibonacci Quarterly*, Dec. 1977, vol. 15, no. 4, pp. 359–61). At the time, we were looking for patterns and generating functions for columns, row sums, and rising diagonal sums in Pascal-like triangles. It never occurred to us to consider Nowak's question: Does Rudy's triangle display predictable *falling* diagonal patterns?

It has been a delight to return to these triangles in search of these other patterns. To help in this matter, we rewrite Rudy's triangle (see **table 1 [Priest and Smith]**), using appropriate multiples of the binomial coefficients—in reverse order. Using this table, we can straightforwardly conjecture and inductively prove that the kth diagonal sequence is given by

$$D_k(n) = \left\{ 2^k \binom{n}{k} \right\}_{n=k}^{\infty}$$

for $k = 0, 1, 2, \ldots$.

At this point, we could not resist enlarging on Nowak's inquiry by also seeking generating functions for these diagonals. To our surprise, the generating functions for $D_k(n)$ were given by

$$\left\{ \frac{2^k}{(1-x)^{k+1}} \right\}$$

for $k = 0, 1, 2, \ldots$, which are more nearly akin to column generators for the simple Pascal triangle than for Rudy's triangle. Loosely speaking, a generating function for the sequence

$$(a_n)_{n=0}^{\infty}$$

is an expression that can be formally expanded (via McLaurin or otherwise) into the series $a_0 + a_1x + a_2x^2 + \cdots$. For example, $1/(1-x)$ is a generating function for the sequence $\{1, 1, 1, \ldots\}$. One finds many applications for generating functions when studying infinite sequences.

We will use column-generating functions (see **table 1 [Priest and Smith]**) to address Nowak's question: What if we change the recursion from upper-left + 2 • upper-right to instead upper-left + 3 • upper-right and make the left border powers of 3? To answer this, we simply place plus symbols between each term of the sequence of column generators

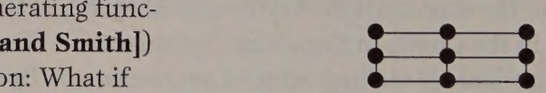
$$\left\{ \frac{x^k}{(1-2x)^{k+1}} \right\}_{k=0}^{\infty}$$

for Rudy's triangle, producing a geometric series whose sum is $1/(1-3x)$. Thus, we obtain the generator for the column of row sums of Rudy's triangle, and it can be used to generate the left border of the conjectured triangle in Nowak's query. Similarly, we can obtain infinitely many such triangles whose left borders can be generated by $1/(1-4x)$, $1/(1-5x)$, ... Thus, we now have infinitely many triangles subject to the "predictable diagonal patterns" question—all providing some additional research material for students.

We now turn to the question, Does Rudy's triangle have a geometric interpretation? For us, this question (and the infinitely many associated with it) was a challenge over a long period of time. As is frequently the case, one of our students, Steve Garrett, produced the answer (see "Hypercubes, Hyperwindows, and Hyperstars," *MT* Oct. 1977, vol. 70, no. 7, pp. 606–9).

Suffice it to say, Rudy's triangle can be interpreted in the following manner: a single dot for zero dimensions; a line segment containing three dots or vertices

for one dimension (think of a straight angle produced by the three dots); and a window containing nine dots joined by six line segments



for two dimensions. At this point, Garrett took three of these windows, joined their twenty-seven dots with the obvious nine line segments, and produced a rectangular parallelepiped containing nine windows. He preferred to call this new object a *hyperwindow*, even though it was only in dimension 3. For obvious reasons, hyperwindows in dimension 4 and higher were never drawn.

Now that we have a geometric interpretation for Rudy's triangle, it is easy to discover geometric interpretations for the infinitely many other Rudy- or Pascal-like triangles. Indeed, at the time

We appreciate the interest and value the views of those who write. Readers commenting on articles are encouraged to send copies of their correspondence to the authors. For publication: All letters for publication are acknowledged, but because of the large number submitted, we do not send letters of acceptance or rejection. Letters to be considered for publication should be in MS Word document format and sent to **mt@nctm.org**. Letters should not exceed 250 words and are subject to abridgment. At the end of the letter include your name and affiliation, if any, including e-mail address, per the style of the section.

Table 1 (Priest and Smith)

	Dimension (n)	Vertices	Edges	Faces	Three-Dimensional	Four-Dimensional	Five-Dimensional	Six-Dimensional	Row Sum
$D_0(n)$									
$D_1(n)$	0	$2^0\binom{0}{0}$							1
$D_2(n)$	1	$2^1\binom{1}{1}$	$2^0\binom{1}{0}$						3
$D_3(n)$	2	$2^2\binom{2}{2}$	$2^1\binom{2}{1}$	$2^0\binom{2}{0}$					9
$D_4(n)$	3	$2^3\binom{3}{3}$	$2^2\binom{3}{2}$	$2^1\binom{3}{1}$	$2^0\binom{3}{0}$				27
$D_5(n)$	4	$2^4\binom{4}{4}$	$2^3\binom{4}{3}$	$2^2\binom{4}{2}$	$2^1\binom{4}{1}$	$2^0\binom{4}{0}$			81
$D_6(n)$	5	$2^5\binom{5}{5}$	$2^4\binom{5}{4}$	$2^3\binom{5}{3}$	$2^2\binom{5}{2}$	$2^1\binom{5}{1}$	$2^0\binom{5}{0}$		243
	6	$2^6\binom{6}{6}$	$2^5\binom{6}{5}$	$2^4\binom{6}{4}$	$2^3\binom{6}{3}$	$2^2\binom{6}{2}$	$2^1\binom{6}{1}$	$2^0\binom{6}{0}$	729
Column Generator		$\frac{x^0}{(1-2x)^1}$	$\frac{x}{(1-2x)^2}$	$\frac{x^2}{(1-2x)^3}$	$\frac{x^3}{(1-2x)^4}$				$\frac{1}{1-3x}$

of Garrett's discoveries, such simple answers were disappointing to many, who had sought geometric interpretations involving triangles, tetrahedrons, Platonic solids, and so on. Along with Nowak we say, there is still ample opportunity for further explorations in search of other geometric interpretations.

Dean B. Priest
dpriest@harding.edu
Steve Smith
ssmith@harding.edu
Harding University
Searcy, AR, Dec. 13, 2011

MORE ON SIGN LANGUAGE

In reference to Doug Houser's comment (Reader Reflections, *MT* Dec. 2011/Jan. 2012, vol. 105, no. 5, pp. 327–28): We all have had students say to us: "Just tell me what the – sign means, and I can do the problem."

How do we know what the – sign means? We just know. Generally, we learn from experience, lots of examples, reading in context. But students don't like that answer.

One year when I was teaching a prealgebra class, I had a very persistent student who was not going to let me get away with "You just know"" or "Practice enough, and you will get it." On every problem, she would ask, "How do you know? Aren't there any rules?" I told her that I had not seen any but that I would think about it and get back to her. In effect, she had given me some homework—a challenge.

Here are the three rules that I came up with. They must be performed in order.

Rule 1: The first sign between two numbers always tells you what to do with the numbers. It is an operation: add, *subtract*, multiply, or divide.

Rule 2 (after identifying all subtraction signs): A sign directly in front of a number means *negative*.

Rule 3: All other – signs mean *the opposite of*.

I demonstrate how to apply the rules by going step by step through an example (see **fig. 1 [Wallace-Gomez]**).

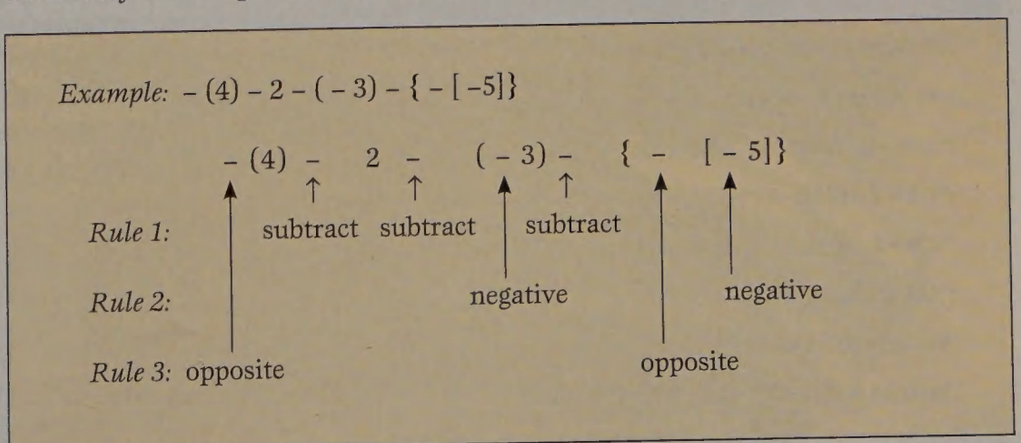


Fig. 1 (Wallace-Gomez)

Then I let students work two more examples (that is usually enough practice). Later, when they have to evaluate algebraic expressions for given values of the variables, I have them put parentheses around each individual variable before substituting the value, no matter whether the value is positive or negative. This step helps with both understanding the meaning of the – signs and with the correct order of operations. Evaluating $3x^2 - x + 5$ for $x = -2$ becomes $3(x)^2 - (x) + 5$ and then $3(-2)^2 - (-2) + 5$.

In this form, students understand the meaning of the signs, and they *can* do the problems.

Patricia Wallace-Gomez
patricia.w.gomez@hotmail.com
Dr. T. J. Owens Gilroy Early College Academy
Gilroy, CA, Jan. 5, 2012

PROBLEM 14, FEBRUARY 2012 CALENDAR

Problem 14 of the February 2012 Calendar (*MT* Dec. 2011/Jan. 2012, vol. 105, no. 5, p. 360) has two holes in its graph. The holes appear where the graph crosses the line whose equation is $y = x$. Changing one or both restrictions to include the equals sign will cover the two holes. If the equations are rotated 45 degrees and then graphed, the resulting heart will be in its more conventional orientation. The two rotated equations are $5x^2 + 6xy + 5y^2 = 200$ where $x \leq 0$ and $5x^2 - 6xy + 5y^2 = 200$ where $x > 0$. If each relation is written as a function of x and then graphed, using a graphing calculator such as a TI-83/84, the resulting graph has gaps on either side of the heart. These gaps can be covered by using the **Shade(** command. A TI-83 program accomplishing this can be seen in **figure 1 (Mitchell)**.

Charles W. Mitchell Jr.
cmitchell57@cox.net
New Trier High School, Winnetka, IL (retired)
Goodyear, AZ, Jan. 17, 2012

PROBLEM 21, OCTOBER 2011 CALENDAR

Re problem 21, October 2011 Calendar (*MT* Sept. 2011, vol. 105, no. 2, p. 121): Why couldn't the children have been twins age 12? The mother would then have been born in 1872 and would have been age 69 in 1941. She would have been 57 when the twins were born. According to Mail Foreign Service, a 70-year-old Indian holds the record for oldest woman to give birth.

Steve Cotter
scotter@sc.rr.com
The Citadel
Charleston, SC, Jan. 22, 2012

PROBLEMS 1, 10, AND 26, MARCH 2012 CALENDAR

Following are alternate solutions to problems 1, 10, and 26 in the March 2012 Calendar (*MT* Feb. 2012, vol. 105, no. 6, pp. 440, 441).

Problem 1

The two-digit numbers can be expressed in a 9×10 matrix:

10	11	12	…	19
20	21	22	…	29
30	31	32	…	39
⋮	⋮	⋮		⋮
90	91	92	…	99

The answer is represented by the lower-left triangular matrix:

10				
20	21			
30	31	32		
⋮	⋮	⋮		
90	91	92	…	98

The total of such two-digit numbers is $1 + 2 + 3 + \cdots + 9 = 9(1 + 9)/2 = 45$.

Problem 10

The x-intercepts of the graph are $\pm 2/\sqrt{a}$. The semidiagonals of the square are of equal length, so $4 = 2/\sqrt{a}$, and $a = 1/4$.

Problem 26

Let $P = (x, y)$. Then $|PA| = 2\,|PB|$, and

$$PA^2 = 4PB^2$$

$$(x+5)^2 + y^2 = 4\left[(x-3)^2 + y^2\right] \rightarrow$$

$$x^2 + y^2 - \frac{34}{3}x + \frac{11}{3} = 0 \rightarrow$$

$$\left(x - \frac{17}{3}\right)^2 + y^2 = \left(\frac{16}{3}\right)^2.$$

J. Sriskandarajah
JSriskandarajah@matcmadison.edu
Madison College
Madison, WI, Feb. 13, 2012

PROBLEM 23, MARCH 2012 CALENDAR

All the solutions to problems 23 and 27 of the March 2012 Calendar (*MT* Feb. 2012, vol. 105, no. 6, p. 441) were great. Here's another alternate solution for problem 23.

The area of parallelogram $DQBS = b \cdot h$ where $b = BS$ and $h = KL$, a side of square $KLMN$. Since BS is the hypotenuse of $\triangle CBS$, the Pythagorean theorem will result in $BS = \sqrt{5}$. The area of $DQBS$ = area of $ABCD$ – area of $\triangle ADQ$ – area of $\triangle CBS$. The area of $ABCD = 4$. The area of $\triangle ADQ = (1/2) \cdot (1) \cdot (2) = 1$. The area of $\triangle CBS$ also equals 1. The area of $DQBS = 4 - 1 - 1$, or 2. The area of parallelogram $DQBS = (BS) \cdot (KL)$. Then $2 = \sqrt{5} \cdot (KL) \rightarrow KL = 2/\sqrt{5}$. The area of square $KLMN = KL^2 = (2/\sqrt{5})^2 = 4/5$ in.^{2}.

Patricia Klagholz
mathpat@live.com
Philadelphia Public Schools (retired)
West Chester, PA, Feb. 9, 2012

```
:ClrHome:Simul:AxesOff:ClrDraw
:-14->Xmin:14->Xmax:-15->Ymin:15->Ymax
:"(-3x+2√(250-4x²))/3/(x≤0)"->Y1
:"(-3x-2√(250-4x²))/3/(x≤0)"->Y2
:"(3x+2√(250-4x²))/3/(x≥0)"->Y3
:"(3x-2√(250-4x²))/3/(x≥0)"->Y4
:Shade(Y2,Y1):Shade(Y4,Y3)
:Text(56,9,"HAPPY VALENTINE'S DAY")
```

Fig. 1 (Mitchell)

BACK FROM LONDON?

In "Linear Reasoning about Circles" (Mathematical Lens, *MT* March 2012, vol. 105, no. 7, p. 494), photograph 2 is identified as the Olympics logo displayed in Penang, Malaysia. Yes, photograph 2 shows five rings, and, yes, the colors are somewhat right, but a closer look reveals major differences with the symbol on the Olympics flag.

First, interlocking rings are a key visual aspect of the Olympic logo since 1920, but these rings do not interlock. Second, the approved Olympic symbol has three rings on top and two on the bottom. My initial thought was that maybe the picture was upside down, so I tried reflecting the image (or rotating it 180 degrees). But even so the colors are still not the traditional blue-yellow-black-green-red from left to right. I thought perhaps this arrangement could be some local combinatorial (5! available) variation, because it certainly made me think Olympics, just as it did for the department editors.

Is there a mathematics educator in Penang who subscribes to *MT* and can solve the mystery of what this image is?

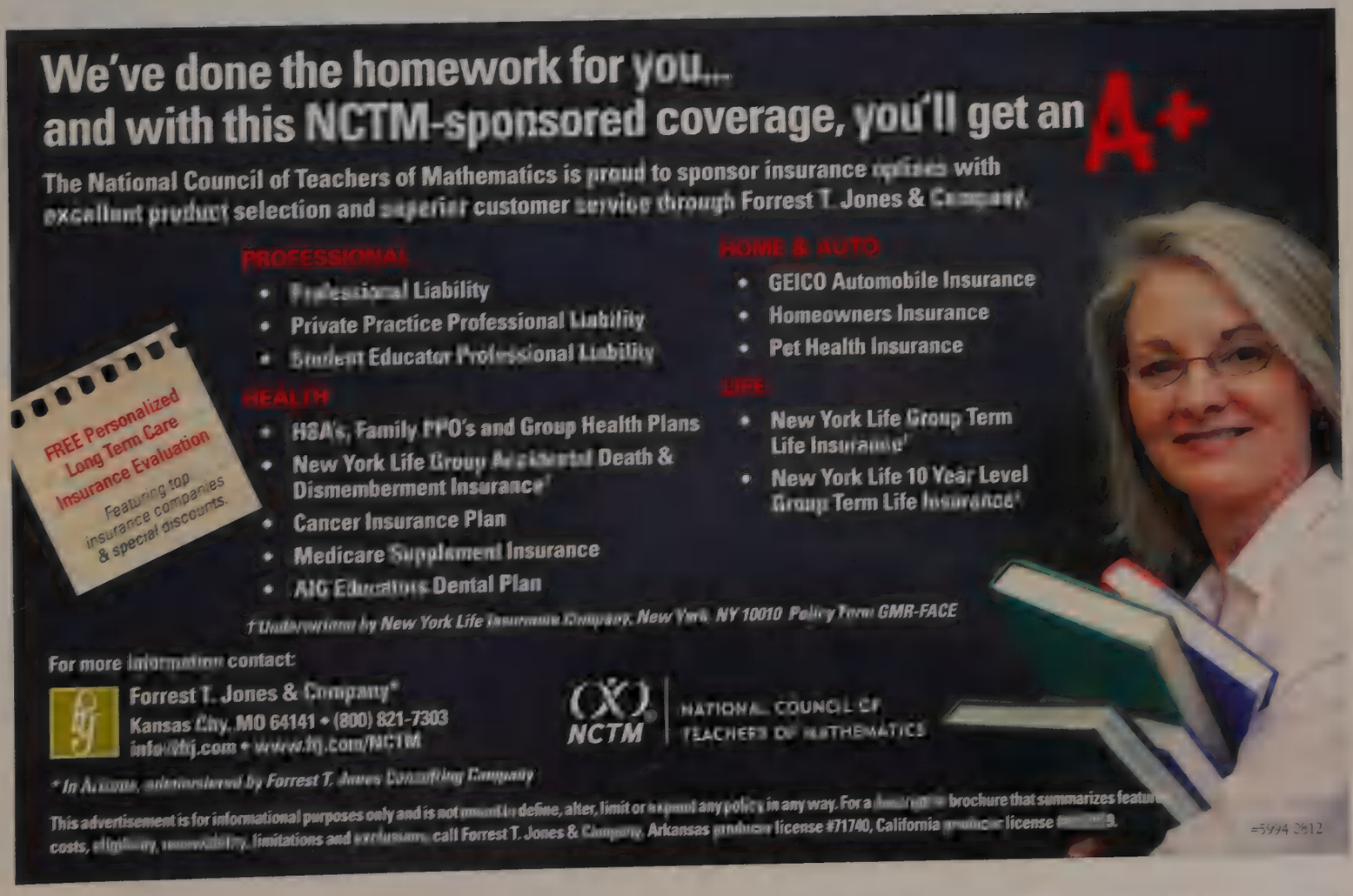
William Jamski
williamjamski@msn.com
Indiana University Southeast
New Albany, IN, Mar. 6, 2012

BIBLICAL MATH REDUX

Gerald Rising (Reader Reflections, *MT* April 2012, vol. 105, no. 8, pp. 566–67) is not the first to use the interesting context of the Bible to discuss significant digits. In *Chance* (fall 1997), David Salsburg explores the biblical census in Numbers 2–4 and wonders whether the figures are rounded to the nearest hundred. After noting that a statistical test shows that the distribution of digits in the hundreds place (but not in the thousands place) differs significantly from a uniform distribution, Salsburg suggests that "what we might be seeing here is an echo of a real census that had once taken place ... but in the process of transmittal, the hundreds digits may have become subject to a slight digit preference with 4s, 5s, and 6s replacing values near 0 and 9" (p. 47).

Back to Rising's specific example of what value of π is implied by I Kings 7:23: Readers may enjoy what I relate in *Journal of Mathematics and Culture* (2006): "Consider the tank's thickness—given three verses later as one-handbreadth, so the inner diameter is 10 cubits minus 2 handbreadths.... Using the Talmudic value of 1/6 cubit for one handbreadth, the inner diameter becomes 9 2/3 cubits and dividing 30 by 9 2/3 yields more accuracy (error: 1.2%)." That passage is followed by a more advanced approach yielding the approximation 333/106 (error: 0.0026%).

Lawrence M. Lesser
Lesser@utep.edu
The University of Texas at El Paso
Mar. 28, 2012


We've done the homework for you...
and with this NCTM-sponsored coverage, you'll get an A+
The National Council of Teachers of Mathematics is proud to sponsor insurance options with excellent product selection and superior customer service through Forrest T. Jones & Company.
PROFESSIONAL
Professional Liability
Private Practice Professional Liability
Student Educator Professional Liability
HEALTH
HSA's, Family PPO's and Group Health Plans
New York Life Group Accidental Death & Dismemberment Insurance†
Cancer Insurance Plan
Medicare Supplement Insurance
AIG Educators Dental Plan
HOME & AUTO
GEICO Automobile Insurance
Homeowners Insurance
Pet Health Insurance
LIFE
New York Life Group Term Life Insurance†
New York Life 10 Year Level Group Term Life Insurance†
FREE Personalized Long Term Care Insurance Evaluation
Featuring top insurance companies & special discounts.
† Underwritten by New York Life Insurance Company, New York, NY 10010 Policy Form GMR-FACE
For more information contact:
Forrest T. Jones & Company*
Kansas City, MO 64141 • (800) 821-7303
info@ftj.com • www.ftj.com/NCTM
NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
* In Arizona, administered by Forrest T. Jones Consulting Company
This advertisement is for informational purposes only and is not meant to define, alter, limit or expand any policy in any way. For a brochure that summarizes features, costs, eligibility, renewability, limitations and exclusions, call Forrest T. Jones & Company. Arkansas producer license #71740, California producer license #0B59994 2012

MEDIA clips

Kristyn K. Wilson // Chris Achong

The Burglar in the Suitcase

Spanish police have arrested a man said to have smuggled himself in a case on to Barcelona airport coaches in order to burgle the luggage compartments.

An alleged accomplice was also arrested after the first man was found "doubled up almost like a contortionist" inside the case, police said....

"Once the trip began, he would get out of the suitcase, search for valuable objects and hide them in a smaller bag he carried with him," regional Catalan police were quoted as saying....

But last Friday, ... a coach company employee noticed a man struggling to put a heavy suitcase in the hold of a coach and alerted the authorities....

Inside, they found the alleged burglar, who was pouring with sweat. His alleged accomplice had already boarded the coach.

The two men, both Polish citizens, were arrested for the theft of a laptop and a GPS device, while police are seeking to trace other possible burglary victims....

Source: "Man 'Hid in Case to Burgle' Barcelona Airport Coach," BBC News EUROPE, 8 June 2011, http://www.bbc.co.uk/news/world-europe-13694382

To answer the following questions, use the diagram in **figure 1 ("The Burglar in the Suitcase")**. Assume that the man's height is 5 ft. 10 in.

1. (*a*) Have someone measure your height (*h*) and the following distances: *a*, from the top of your head to your hips; *b*, from your hips to your knees ; and *c*, from your knees to your feet. Use your measurements as a reference to determine the following ratios for the burglar in the suitcase: h/a, h/b, and h/c.

   (*b*) What assumptions have you made?

2. (*a*) How deep would the suitcase need to be for the man to fit inside?

   (*b*) Using that depth, find the volume of the suitcase.

3. According to your calculations from parts 1 and 2, would the man be able to fit in the suitcase? If so, how much room was there inside the suitcase and above his head? Use calculations to support your conclusion.

4. Research the luggage restrictions for some airlines. Would the burglar's suitcase meet the restrictions or be flagged? Justify your answer.

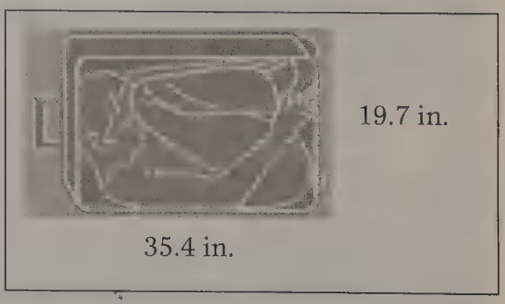


Fig. 1 ("The Burglar in the Suitcase")

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit http://www.nctm.org/publications/content.aspx?id=10440#media.

Edited by ***Louis Lim****, louis.lim1@gmail.com*
Thornhill Secondary School, Thornhill, ON, Canada

Lionel Garrison*, garrison@horacemann.org*
Horace Mann School, Bronx, NY

Climate History in an Ice Core

Earth's polar temperature has swung wildly—by as much as 15 degrees Celsius (27 degrees Fahrenheit)—over the last 800,000 years, an Antarctic ice core has revealed. In 2004 scientists led by Jean Jouzel of the Laboratory of Climate and Environmental Sciences (LSCE) in Gif-sur-Yvette, France, pulled up the final chunk of ice core from a drill hole in the center of Antarctica. At 3,260 meters (10,695 feet) long, drilling the ice core was a marathon effort in one of the most hostile and remote places on Earth....

Although it is not the deepest Antarctic core..., its compressed ice does provide the longest polar climate record, going back 800,000 years....

Measuring deuterium, a form of the element hydrogen, enabled the team to piece together the temperature record in the ancient ice. The new climate record covers an additional cycle of glacial change, amounting to 11 cycles in total, lead author Jouzel said. Plugging the data from the entire core into an atmospheric model, the scientists were able to reconstruct a reliable temperature record for the past 800,000 years.

... the team showed that the coldest period occurred around 20,000 years ago, during the last glacial maximum, when the ice sheets were at their peak. It was about 10 degrees Celsius (18 degrees Fahrenheit) colder than today.... Meanwhile, the warmest period was during the last interglacial period, which is an interval of warmer global average temperature that separates ice ages. At that time, around 130,000 years ago, it was a balmy 4.5 degrees Celsius (8.1 degrees Fahrenheit) warmer than today....

Source: Kate Ravilious, "New Ice Core Reveals 800,000 Years of Climate History," *National Geographic News*, July 5, 2007, http://news.nationalgeographic.com/news/2007/07/070705-antarctica-ice.html. © 1996-2008 National Geographic Society. All rights reserved.

1. Using today's global average temperature of 60.8°F, determine the following:

   (*a*) the lowest global average temperature in the last glacial maximum

   (*b*) the highest global average temperature during the last interglacial period

2. Assuming that the cycles of glacial change follow a regular periodic pattern, determine the period and the amplitude of the model that represents the glacial cycle.

3. Will the next critical point be a local maximum or a local minimum? Is the present-day global average temperature on an increasing part of the cycle or a decreasing part of the cycle? Provide reasons for your answer.

4. Derive a sinusoidal equation that represents the glacial cycles with temperature (T) as a function of time (t).

5. The clip states that 11 glacial cycles occurred in the last 800,000 years. Does your model support this assertion? Explain using mathematical reasoning.

6. Draw a sketch of one cycle of your sinusoidal function that includes information in the clip. Be sure to include a scale and label all important information.

Michigan State University
is accepting applications for its
Doctoral Program in Mathematics Education
Designed for those who show promise of becoming researchers and leaders in state, national, and international mathematics education communities. Administered jointly by the Colleges of Natural Science and Education, this program prepares researchers to address critical issues in mathematics teaching, learning, curriculum and policy. Assistantships/fellowships available!
Apply by December 1 for enrollment the following fall!
http://prime.msu.edu
Lisa Keller, Program Coordinator
517.432.5472, kellerl@msu.edu

MEDIA CLIPS solutions

"The Burglar in the Suitcase" answers

1. (*a*) Answers will vary. For the author, $h = 65$, $a = 35$, $b = 12$, and $c = 18$. We solve the following proportions (all measurements in in.):

$$\frac{65}{35} = \frac{70}{a} \rightarrow a \approx 37.7$$
$$\frac{65}{12} = \frac{70}{b} \rightarrow b \approx 12.9$$
$$\frac{65}{18} = \frac{70}{c} \rightarrow c \approx 19.4$$

(*b*) Answers will vary, but one important assumption that the author makes is that the man in the suitcase has the same proportions that she does.

2. (*a*) To determine the minimum depth of the suitcase, hypothesize using your own measurements about what the minimum width of the man's body could be. The author considered her shoulder width (18 in.) as the minimum. The man would need a minimum depth of 18 in. to allow for the breadth of his shoulders (assuming that he is rounding them forward).

(*b*) The suitcase would have a volume of around 12,552.8 in.^{3}:

$$V = l \cdot w \cdot h$$
$$= (35.4)(19.7)(18)$$
$$= 12{,}552.84 \text{ in.}^3$$

3. The man probably was able to fit because he rolled his neck forward and shortened himself by approximately 4 in., enabling him to have about 1.8 in. above his head.

4. Each student might research a different airline. The maximum weight allowed for most airlines is 50–70 lb. Most grown men weigh more than 70 lbs., so a suitcase with a man inside would certainly be flagged. For example, Spanair (http://portal.spanair.com/web/en-gb/Luggage/) has a limit of 32 kg. Compute as follows: 1 kg = 2.2 lb.; therefore, 32(2.2) = 70.4 lb.

Many airlines also have restrictions on the sum of the linear dimensions of a piece of luggage. Note that this limit is not the girth of the luggage; many shipping companies have set restrictions on the maximum girth (distance around a cross section) of a package. For Spanair, the maximum linear dimension is 158 cm. The linear dimensions of the given suitcase are $35.4 \times 19.7 \times d$, but these dimensions are in inches. To change to centimeters, we use the conversion factor 1 in. = 2.54 cm, so $(35.4 + 19.7) \cdot 2.54$ cm/in. $+ d$ cm $\approx 140 + d$ cm. Thus, $d \approx 18$ cm ≈ 18 in./2.54 cm/in. ≈ 7.1 in. Since 7.1 in. is too narrow for the man's shoulders, he would not fit in an allowable suitcase.

"Climate History in an Ice Core" answers

1. The present-day global average temperature is 60.8°F.

(*a*) The global average temperature 20,000 years ago was 18°F colder. Therefore, it was 60.8° – 18.0° = 42.8°F.

(*b*) The global average temperature 130,000 years ago was 8.1°F warmer. Therefore, it was 60.8° + 8.1° = 68.9°F.

2. The maximum and minimum temperatures both occurred during the most recent glacial cycle. As a result, the time elapsed between the maximum and minimum temperatures represents half the period, P. So $P/2 = 130{,}000 - 20{,}000 \rightarrow P = 220{,}000$ years.

Amplitude, a, is given by the equation $a = (\text{maximum} - \text{minimum})/2 = (68.9° - 42.8°)/2 = 13.05°$F. The clip also states that there is a change of temperature of 27°F between maximum and minimum temperatures, so 13.5°F would also be an acceptable answer.

3. The next critical point will be a local maximum. The highest temperature occurred during the last interglacial period, meaning that the last maximum occurred 130,000 years ago. The lowest temperature occurred during the last glacial period, meaning that the previous minimum was 20,000 years ago. Since the last minimum was most recent in history after the last maximum, the next critical point must be a maximum. The present-day temperature is on an increasing part of the function.

4. A sinusoidal function is defined by $y = a \cdot f(k[x - d]) + c$ where a is the amplitude, $k = 2\pi/P$, d is the horizontal phase shift, c is the vertical translation, and f represents either the sine or cosine function.

From part 2, we know that $a = 13.05$ and $P = 220{,}000$, so $k = 2\pi/220{,}000 = \pi/110{,}000$. Further, we know that $c = (68.9 + 42.8)/2 = 55.85°$ F. Since the previous critical point was a minimum, we opt to use a negative cosine function (reflecting the curve over the sinusoidal axis). The phase shift is 20,000 to the left. One possible function to represent the glacial cycles follows:

$$T = -13.05\cos\left(\frac{\pi}{110{,}000}\left(t + 20{,}000\right)\right) + 55.85°\text{F}$$

5. If 11 glacial cycles occur in 800,000 years, one cycle is $800{,}000/11 \approx 72{,}727$ years. This period does not correlate to the period of the sinusoidal function. The model is incorrect for the following reasons. The maximum and minimum points repeat in a cycle but not in regular intervals. We know this because the last glacial cycle had a period of 220,000 years. On the basis of this period, 11 cycles should result in an ice core that had data for $11 \cdot 220{,}000 = 2{,}420{,}000$ years, much longer than what we are told in the clip. The only way to

make sense of the information is that the last glacial cycle must have been longer than other glacial cycles in the 800,000 years. The clip also states that the highest and lowest global average temperatures in the last glacial cycle were the highest and lowest recorded to date, implying that the previous cycles had milder maximum and minimum temperatures.

According to this analysis, the glacial cycles cannot be represented using only the data in the clip; further, the assumption that the glacial cycles occur in regular intervals is incorrect. The maximum and minimum temperatures are also inconsistent from cycle to cycle. The data given in the clip do not satisfy the criteria for a periodic function. As a result, the sinusoidal function should not be used to model the data.

6. The graph appears in **figure 1 ("Climate History in an Ice Core")**.

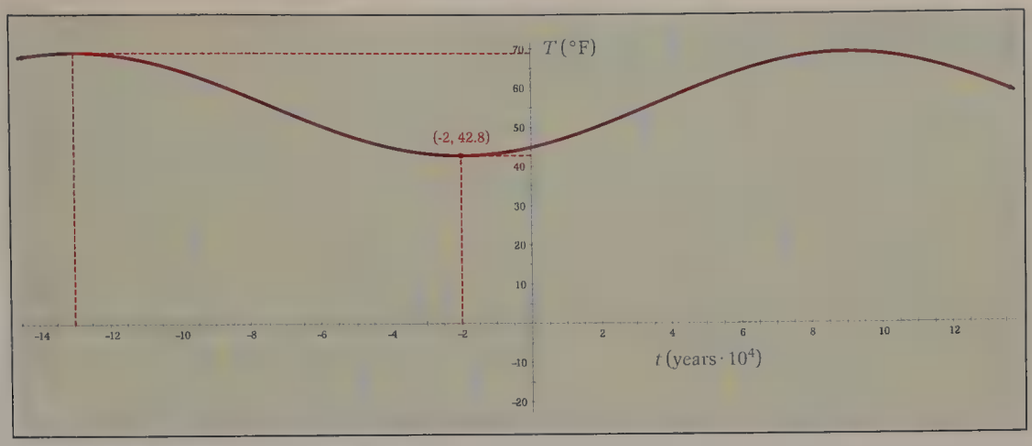


Fig. 1 ("Climate History in an Ice Core")

KRISTYN K. WILSON, wilsonkr@ldrsb.net, who submitted "The Burglar in the Suitcase," teaches mathematics at Ridgetown District High School in Ridgetown, Ontario. She is interested in finding new and unique ways to spark mathematics interest in her students. CHRIS ACHONG, christopher.achong@yrdsb.edu.on.ca, who submitted "Climate History in an Ice Core," teaches mathematics and science at Markville Secondary School in Markham, Ontario, Canada. He has a keen interest in fostering students' abilities to think critically and make connections.

NCTM's Member Referral Program

Let's Add Up! Refer. Receive.

Participating in NCTM's Member Referral Program is **fun, easy, and rewarding**. All you have to do is refer colleagues, prospective teachers, friends, and others for membership. Then as our numbers go up, watch your rewards add up.

Learn more about the program, the gifts, and easy ways to encourage your colleagues to join NCTM at **www.nctm.org/referral**. Help others learn of the many benefits of an NCTM membership—**Get started today!**

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

(800) 235-7566 | WWW.NCTM.ORG

Douglas Wilcock

A Road for Every Wheel

Photograph 1
Tricycle with square wheels

CEM INALTONG

Photograph 2
Cross section of surface (cycloid)

CEM INALTONG

Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Edited by ***Ron Lancaster,*** *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

Can you imagine riding a tricycle with square wheels? Can you imagine that this tricycle would give you as smooth a ride as a traditional tricycle? A *New York Times* article (Chang 2011) described a tricycle that had square wheels but that could be ridden "smoothly around a circular path ridged like a flower's petals." It then explained that the ridged surface on which the tricycle rode undulated such that "the tricycle's axles—and the rider—remain in the same height as they move."

Photograph 1 shows the tricycle with a rider, and **photograph 2** shows a close-up of the riding surface. We want to find the path that the axles take to provide a smooth ride. We will assume that a cross section of the surface is a *cycloid*, the path a point on a circle takes as the circle rolls along a horizontal line (see **figs. 1** and **2**). Many websites have animations showing the creation of a cycloid; mathworld.wolfram.com/cycloid.html is one such site.

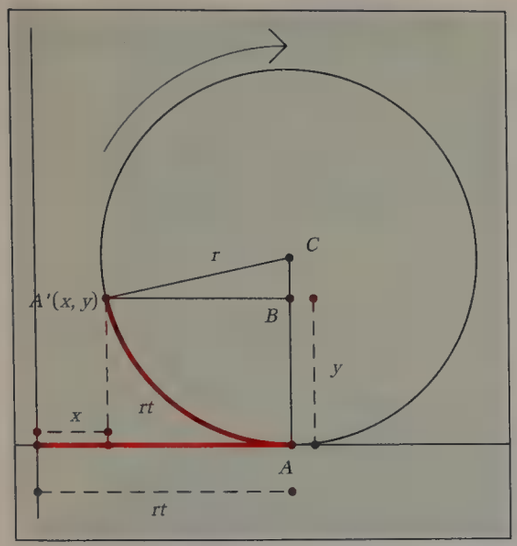


Fig. 1 As the circle rolls to the right, point *A* traces a cycloid.

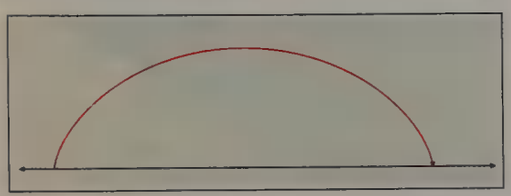


Fig. 2 The first arch of the cycloid is traced by the path of point *A*.

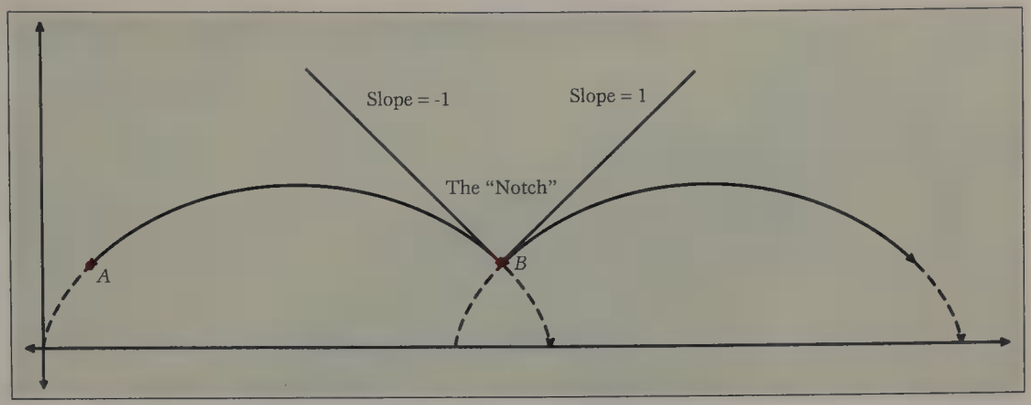


Fig. 3 The square wheel will fit in the notch exactly.

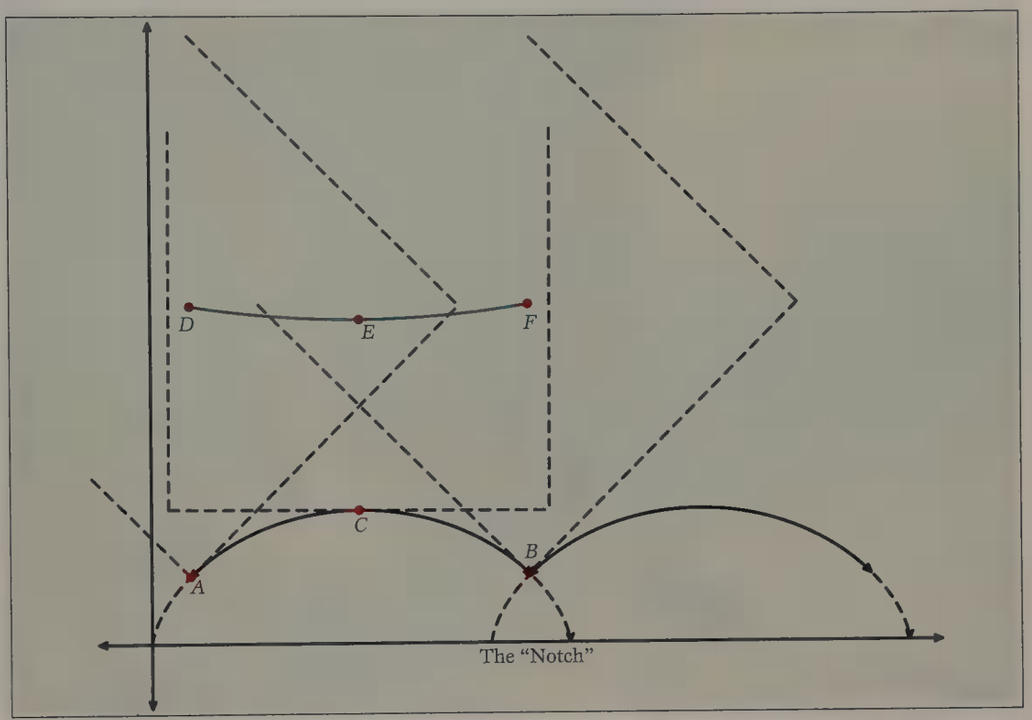


Fig. 4 As the wheel traverses the first arch, the path of the axle is *DEF*.

1. If we assume that the circle does not slip as it travels horizontally and that one rotation of the circle will occur in 2π units of time, there is a direct relationship between angle t and time t. What are the parametric equations for x and y as a function of angle t and radius r?

2. The square wheel of the tricycle will travel from what we will call the "notch" (see **fig. 3**) to the peak of the cycloid and then on to the next notch. At the notch, the lines tangent to the two arches of the cycloid will be perpendicular to each other, allowing the square wheel to fit. Also, a vertical line of symmetry exists at each notch. Hence, the tangent lines will have slopes of 1 and -1 (see **fig. 3**).

   (*a*) Find the derivatives with respect to t for the x and y components of the parametric cycloid equation. What equation represents the slope of the cycloid at any given point?

   (*b*) Point A in **figure 3** will mark the beginning of the tricycle's path on the first arch of the cycloid. If the origin is our frame of reference with time $t = 0$, at what time is the corner of the bicycle wheel at point A? What are the x- and y-coordinates of point A?

   (*c*) Starting at the origin at time $t = 0$, determine at what time the corner of the wheel arrives at the notch (point B). What are the x- and y-coordinates of point B?

3. In **figure 4**, the wheel traverses the first arch of the cycloid. Note that we show only part of the square wheel as it moves along the arch. The axle takes the path of curve DEF, the locus of points formed by the intersection of the square wheel's diagonals as the wheel traverses the cycloid. Find the length of arc ACB.

We have determined the length of a side of the square. In the next question, we reverse the process.

4. Suppose that the tricycle has square wheels that measure 30 cm on a side. Find the radius of the circle that generates the cycloid in which one side of the tricycle wheel creates arc ACB (see **fig. 4**).

5. As stated above, the path of the axle is arc DEF (see **fig. 4**). The surface of the cycloid must be adjusted to make the ride smooth. The undulation that would be required would be a reflection of arc DEF about the horizontal line DF; the surface would be pushed up as the tricycle moves from D to E and then pushed back down as the tricycle moves from E to F. In terms of the radius, r, of the circle generating

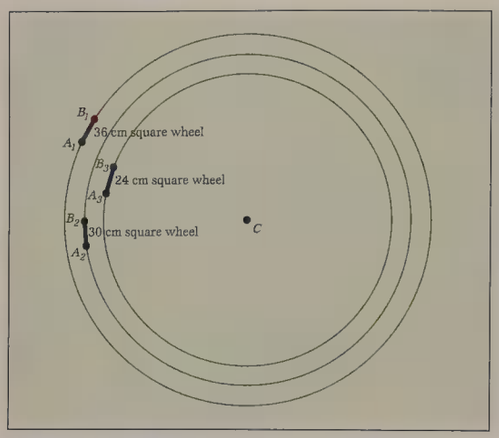


Fig. 5 The bird's-eye view of the bicycle track shows three different sizes of wheels.

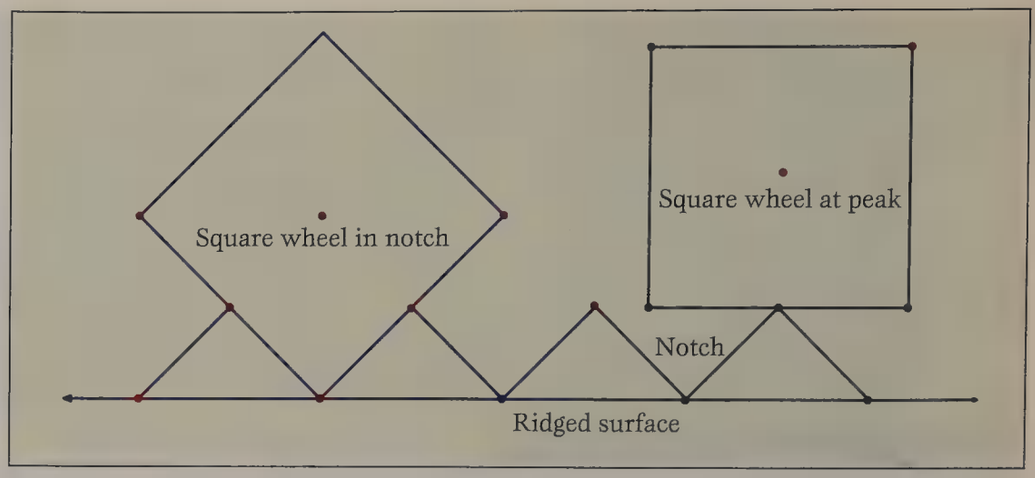


Fig. 6 A sawtooth design is another way to achieve the same effect.

the cycloid and for the specific 30 cm square wheel, how much would the surface have to be pushed up at point C (corresponding to E on the axle path) so that the ride is smooth?

Once the undulation is accomplished, we have a smooth ride. But there is another catch to this tricycle: The wheels are not the same size. If you look closely at **photographs 1** and **2,** you will notice some red lines; these are the "tracks" that the wheels follow.

6. Suppose that the three wheels of the tricycle are 36, 30, and 24 cm; they are shown in **figure 5** as A_1B_1, A_2B_2, and A_3B_3, respectively. If one lap around the track comprises 32 arches, what are the lengths of the three radii (A_1C, A_2C, and A_3C)?

7. In the bird's-eye view shown in **figure 5,** the segment labeled A_iB_i represents the linear segments shown in **figure 4**.

   (*a*) What is the total of all segments AB for the 24, 30, and 36 cm wheel sizes as the tricycle completes one lap of the track?

   (*b*) What is the actual distance the wheels travel as they complete one lap of the track?

   (*c*) Consider the polygons formed by joining all the segments for each wheel of the tricycle as it completes one lap. What is the circumference of the circumcircle that encompasses that polygon?

8. A cycloid base is not the only way to construct this track. **Figure 6** shows a sawtooth track. Describe the path of the tricycle's axle. Assume that the length of a side of the square wheel is 4*s*.

9. What other combinations of wheel design and surface would give interesting results? Work out a design and the mathematics that would accompany your design.

[*Mathematical Lens solutions begin on page 258.*]

BECOME AN ADVOCATE

FOR THE IMPROVEMENT OF MATHEMATICS TEACHING AND LEARNING!

Go to www.nctm.org and click on

- Research, News, & Advocacy
- Advocacy
- Be an Advocate

You will find information on current issues, legislative alerts, how to get your voice heard, how to contact our elected officials, and much more.

Can you see the math?

Bring mathematics to life.

The Geometer's Sketchpad® transforms how students see math. We pioneered *Dynamic Geometry*®—cited by the Common Core State Standards as a strategic tool—and a mathematical metamorphosis ensued in classrooms around the world. Bring math to life by modeling and exploring math's big ideas with *Sketchpad*.

THE GEOMETER'S SKETCHPAD

from Key Curriculum, a McGraw-Hill Education company

See the math at
go.keycurriculum.com/butterfly

MATHEMATICAL LENS solutions

1. Using **figure 1** (p. 255), we have $x = rt - r\sin t = r(t - \sin t)$ and $y = r - r\cos t = r(1 - \cos t)$.

2. (*a*) For the derivatives of the parametric equation, we have

$$\frac{dx}{dt} = r - r\cos t$$
$$= r(1 - \cos t)$$
$$\frac{dy}{dt} = r\sin t$$

and thus

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{r\sin t}{r(1-\cos t)}$$
$$= \frac{\sin t}{(1-\cos t)}.$$

(*b*) At point A, the slope of the tangent line is 1. So,

$$1 = \frac{\sin t}{1-\cos t} \rightarrow$$
$$1 = \sin t + \cos t \rightarrow$$
$$t = \frac{\pi}{2}.$$

At $t = \pi/2$, $x = r(\pi/2 - 1)$ and $y = r$.

(*c*) The wheel arrives at the notch (point B) when the slope is equal to -1 (see **fig. 3,** p. 255):

$$-1 = \frac{\sin t}{(1-\cos t)} \rightarrow$$
$$-1 = \sin t - \cos t \rightarrow$$
$$t = \frac{3\pi}{2}$$

At this point, $x = r(3\pi/2 - \sin(3\pi/2)) = r(3\pi/2 + 1)$, and $y = r(1 - \cos(3\pi/2)) = r$.

3. We set up and evaluate the parametric arc lengths:

$$s = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{\left(r(1-\cos t)\right)^2 + \left(r\sin t\right)^2}\, dt$$
$$= r\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{1 - 2\cos t + \left(\cos t\right)^2 + \left(\sin t\right)^2}\, dt$$
$$= r\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{2 - 2\cdot\cos t}\, dt$$
$$= \sqrt{2}r \cdot \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{1-\cos t}\, dt \approx 5.657r$$

Now we apply the half-angle formula—

$$\sin A = \sqrt{\frac{1-\cos 2A}{2}} \rightarrow$$
$$\sqrt{2}\sin A = \sqrt{1-\cos 2A}$$

—so that our integral becomes

$$s = 2r\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin\frac{t}{2}\, dt$$
$$= -4r\cos\frac{t}{2}\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$
$$= 4\sqrt{2}r.$$

4. To determine r when the wheel is 30 cm, we find that $r = 30/4\sqrt{2} \approx 5.3$ cm.

5. Recall that the side of the square is $4\sqrt{2}r$. The difference, d, in axle heights from the notch to the top of the cycloid (see **fig. 7**) is given by

$$d = 5r - 2r\left(1 + \sqrt{2}\right)$$
$$= r\left(3 - 2\sqrt{2}\right).$$

For the 30 cm wheel, the difference (D) is calculated as follows:

$$D = \frac{30}{4\sqrt{2}}\left(3 - 2\sqrt{2}\right)$$
$$= \frac{45}{2\sqrt{2}} - 15$$
$$\approx 0.910 \text{ cm}$$

6. Since the cycloidal base contains 32 arcs, each of the angles A_1CB_1, A_2CB_2, and A_3CB_3 contains $2\pi/32$ radians, or 11.25°. Since A_2CB_2 is isosceles with $A_2C = CB_2$ and $A_2B_2 = 30$ for the 30 cm wheel, we can find the length of A_2C, the radius R of the circle. We can solve the equation $\sin(\pi/32) = 15/A_2C$; thus, $A_2C = R \approx 153.034$ cm. Or we can use the law of cosines to obtain $30^2 = 2R^2 - 2R^2\cos 11.25°$ to get the same answer. For the 36 cm wheel, $R \approx 183.641$ cm; for the 24 cm wheel, $R \approx 122.428$ cm.

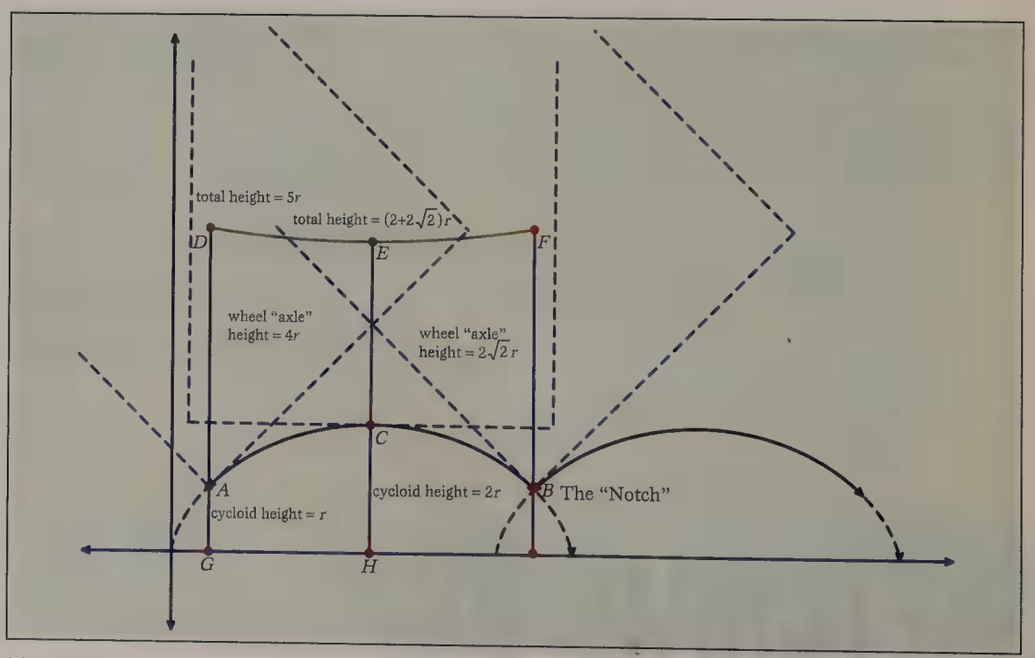


Fig. 7 The maximum height that the wheel's axle will make is 5*r*.

Table 1 Determining Wheel Sizes

	Wheel Size (W/cm)	Explanation
Part (a)	$\frac{W}{4\sqrt{2}}(\pi+2)32$	The difference in the x-coordinates of A and B is $\pi+2$; since W = arc length and $AL = 4r\sqrt{2}$, we use $W/4\sqrt{2}$.
Part (b)	$32W$	
Part (c)	$\frac{\pi W}{\sin\left(\frac{\pi}{32}\right)}$	$\sin\left(\frac{\pi}{32}\right) = \frac{\left(\frac{W}{2}\right)}{R}$ $R = \frac{W}{2\sin\left(\frac{\pi}{32}\right)}$ so $\text{circumference} = 2\pi\left(\frac{W}{2\sin\left(\frac{\pi}{32}\right)}\right)$ $C = \frac{\pi W}{\sin\left(\frac{\pi}{32}\right)}$

Table 2 Approximate Wheel Sizes (nearest cm)

	24 cm. Wheel	30 cm. Wheel	36 cm. Wheel
Part (a)	698 cm	873 cm	1047 cm
Part (b)	768 cm	960 cm	1152 cm
Part (c)	769 cm	962 cm	1154 cm

7. For part (a), the length of segment AB is the difference in the coordinates of B and A found in question 2 and the radius found in question 4. For part (b), the length of arc AB is the side of the square. And for part (c), the radius of the track was found in question 6. A general case with explanation is shown in **table 1**. Specific answers for the given wheel sizes are in **table 2**.

8. **Figure 8** shows the axle of the square wheel at its minimum and maximum heights. Using **figure 9**, we find the difference (d) in height between the notch and the peak of the sawtooth:

$$d = s\left(2+\sqrt{2}\right) - s\left(2\sqrt{2}\right)$$
$$= s\left(2+\sqrt{2}-2\sqrt{2}\right)$$
$$= s\left(2-\sqrt{2}\right)$$

For the 30 cm wheel, $s = 7.5$ cm, and the height difference between the peak of the sawtooth and the notch is approximately 4.39 cm. This difference is far greater than the 0.910 cm undulation for the cycloid base that we calculated in the solution to question 5.

9. It would be interesting to see how a triangular or a pentagonal wheel would work on the cycloid base. Some students might be interested in curves of constant width, known as Reuleaux polygons. For an interesting discussion of Reuleaux triangles that uses The Geometer's Sketchpad® animation, go to http://whistleralley.com/reuleaux/reuleaux.htm.

REFERENCE

Chang, Kenneth. June 28, 2011. "One Math Museum, Many Variables." *New York Times*, Science Times, p. 1.

DOUGLAS WILCOCK, dougwilcock@capecodacademy.org, recently retired as a mathematics, statistics, and economics teacher at Cape Cod Academy in Osterville, Massachusetts. He remains interested in finding mathematical applications from the world around us.

more4U

For a photograph for your own mathematical questions, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt029**.

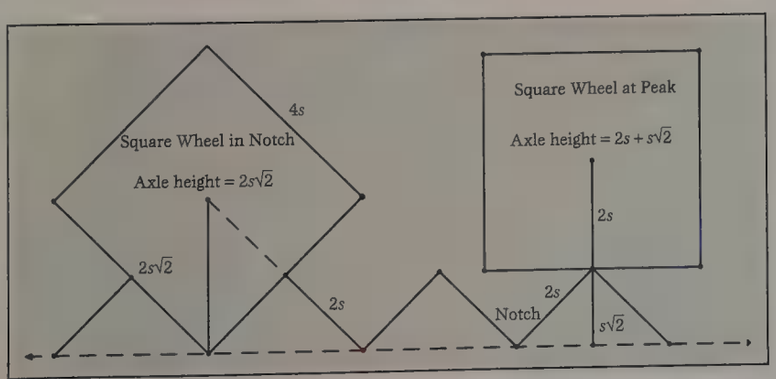


Fig. 8 Axle heights for the sawtooth design show a greater bump than the cycloid design did.

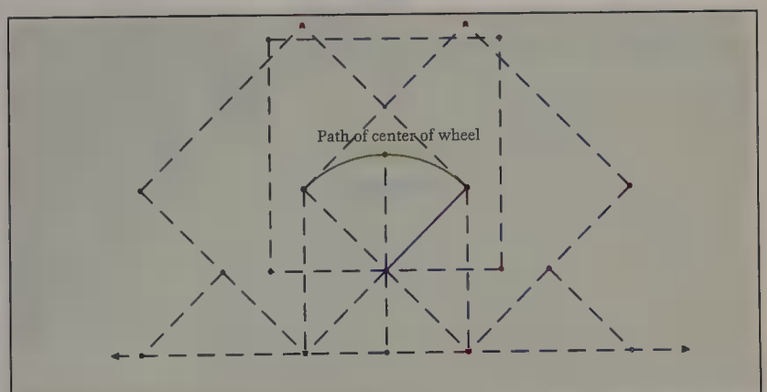


Fig. 9 The path traced by the axle in the sawtooth design produces an arc.

Using
Multiple
Repre

Experience with multiple representations fosters students' robust understanding of what functions are, how they behave, and how they can be composed.

sentations to Teach Composition of Functions

Scott Steketee and Daniel Scher

Functions serve "a central and unifying role" in mathematics (Selden and Selden 1992, p. 1), and many mathematics educators argue that functions should be the central theme around which the secondary school algebra curriculum is organized (e.g., Chazan 2000). Yet many researchers (Chazan and Yerushalmy 2003; Schoenfeld and Arcavi 1988) have noted that concepts of variables and functions are particularly difficult for students to understand, and Oehrtman, Carlson, and Thompson (2008) observe that "students continue to emerge from high school and freshman college courses with a weak understanding" of functions. Most algebra teachers would agree.

Some aspects of functions that students find difficult to understand follow:

- Variables play multiple roles—as unknown constants (as in $2x = 6$); as placeholders for measurements (as in $A = b \cdot h$); or as numbers that do vary (as in $y = x - 2$) (Usiskin 1999; Kieran 2006). These multiple roles are often confusing to students.
- Often the "variables" that students encounter do not vary in any perceptible sense.
- Students may become fixated on particular representations, such as $y = mx + b$ (Cooney, Beckman, and Lloyd 2010, p. 1), and their experiences may suggest that the quantities that do vary are m and b rather than x and y (Selden and Selden 1992, p. 6).
- Students may be confused about the relationship between discrete tables of values and continuous Cartesian graphs.

TIMEGOODWIN/ISTOCKPHOTO

- Cartesian graphs, a powerful tool for more sophisticated students, can be hard to understand for beginning algebra students (Goldenberg, Lewis, and O'Keefe 1992) and are typically of little help in understanding composition, even for more advanced students.
- Students have difficulty describing the behavior of functions (Hazzan and Goldenberg 1996, p. 269) and in thinking of functions as objects in their own right, subject to mathematical manipulation.

A useful strategy for alleviating these difficulties is to provide students ample opportunities to directly and continuously manipulate variables as they observe and analyze the resulting behavior of functions. Such experiences could include a variety of visually effective representations that highlight different aspects of the concept of function and encourage students to develop a fuller concept image (Yerushalmy and Shternberg 2001). Interplay between algebraic and geometric representations can be particularly useful to help students "learn more effectively and remember better" (Banchoff 2008).

Composition of functions is one of the five big ideas identified in NCTM's *Developing Essential Understanding of Functions, Grades 9–12* (Cooney, Beckmann, and Lloyd 2010). Through multiple representations (another big idea) and the use of The Geometer's Sketchpad® (GSP), students can directly manipulate variables and thus see dynamic visual images of function behavior. The object is to foster students' deep and flexible understanding of what functions are, how they behave, and how they can be composed—crucial prerequisites for mastery of related topics such as transformations of functions, families of functions, and inverses of functions.

Depending on students' previous use of GSP and their familiarity with geometric representations of functions, the five activities described in this article will require from three to five class periods of about forty-five or fifty minutes each. Do not rush students; a thorough and unhurried discussion at the end of each activity is critical. Class discussions allow students to think again about what they have done and observed, to review their use of function notation, to compare and contrast different representations of function composition, and to consolidate their understanding. Although all five activities need not be used, doing so gives students the time and experiences that many need to consolidate and generalize their understanding of this challenging topic.

GEOMETRIC REPRESENTATION

The first representation of function composition we use is a geometric one—composing a dilation and a translation.

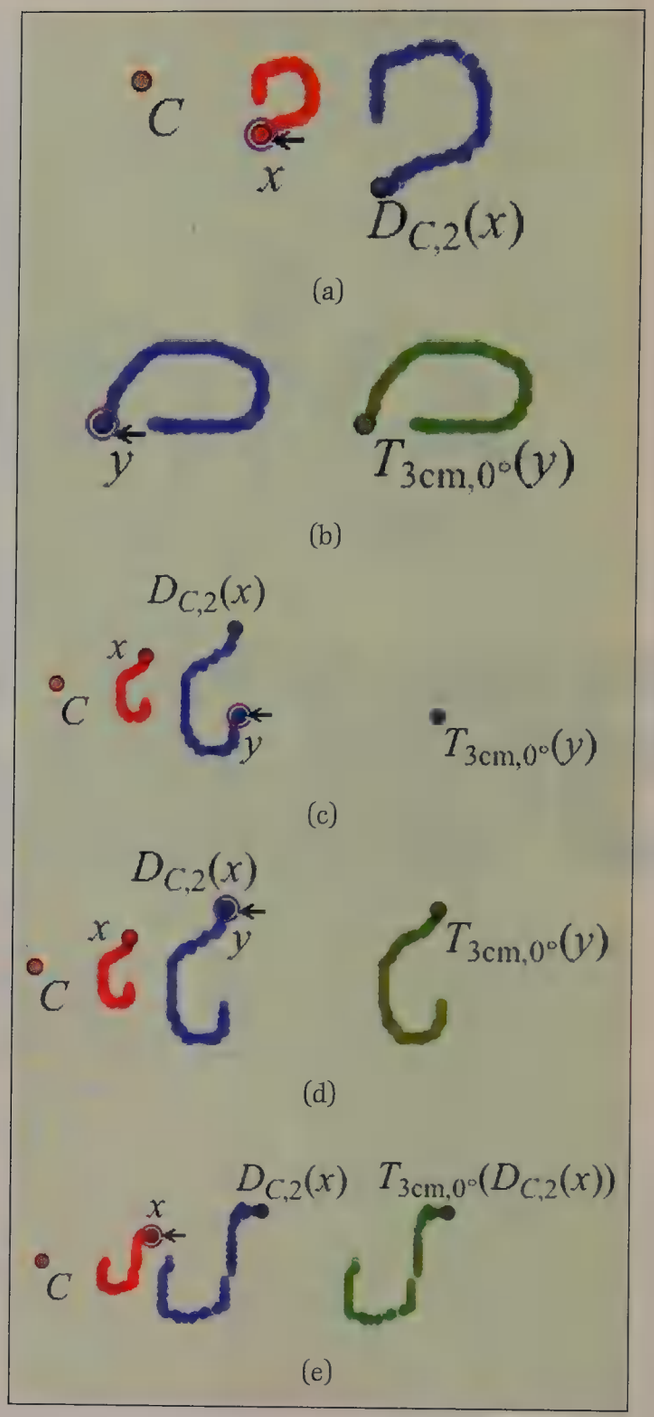


Fig. 1 Students construct and trace a dilation function $D_{C,2}$ (**a**) and a translation function $T_{3cm,0°}$ (**b**); create a new trace of $D_{C,2}$ (**c**); drag the independent variable of $T_{3cm,0°}$ to follow the trace of the dependent variable of $D_{C,2}$ (**d**); and then merge the independent variable of $T_{3cm,0°}$ to the dependent variable of $D_{C,2}$ and trace again (**e**).

Students begin by dilating point x by a scale factor of 2 about point C and labeling the dependent point $D_{C,2}(x)$ to describe its construction. By dragging independent variable x, students can directly observe the behavior of the function. By tracing both independent variable x and dependent variable $D_{C,2}(x)$ (see **fig. 1a**), students create a picture of the behavior—a geometric equivalent of a table of values but one that shows the behavior of the function visually, in a clearer and more compelling way than is possible with a table of values.

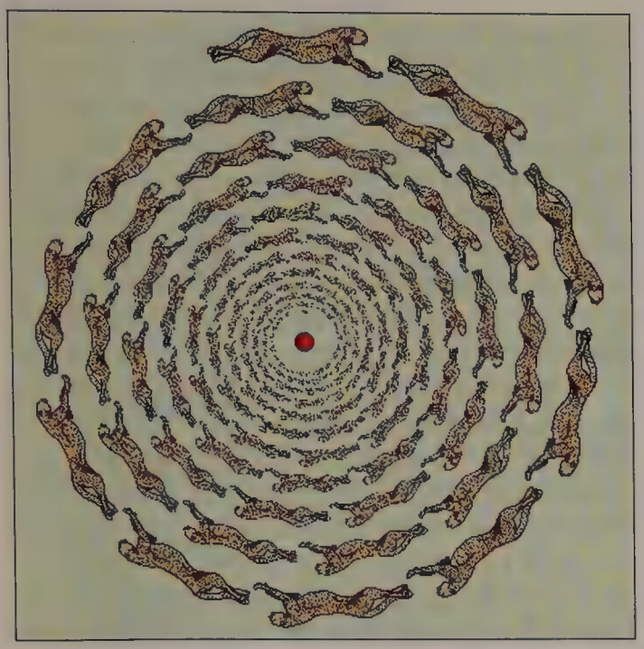


Fig. 2 Students repeatedly apply a composed function to a picture.

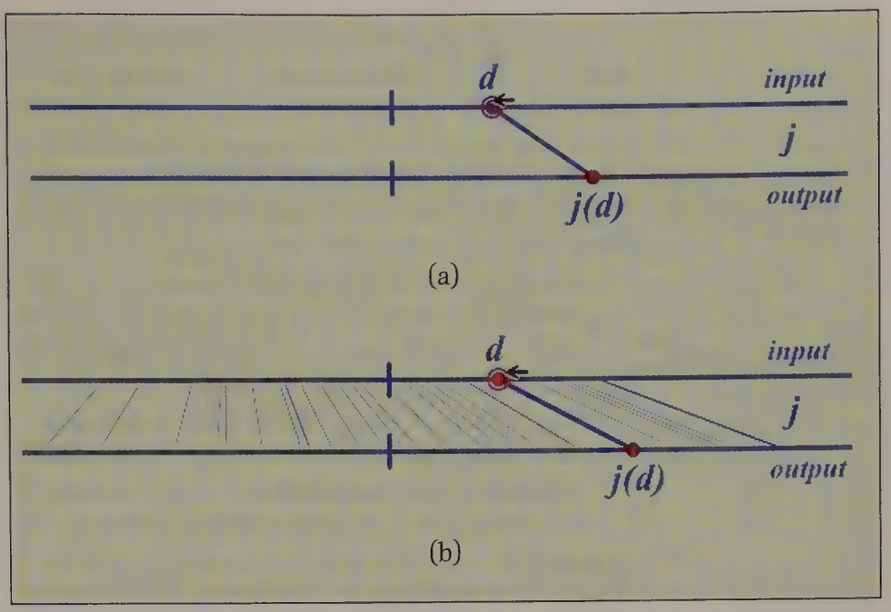


Fig. 3 Students drag independent variable d to observe the behavior of dependent variable $j(d)$ (**a**) and trace the connecting segment to reveal the correspondence between the domain and the range (**b**).

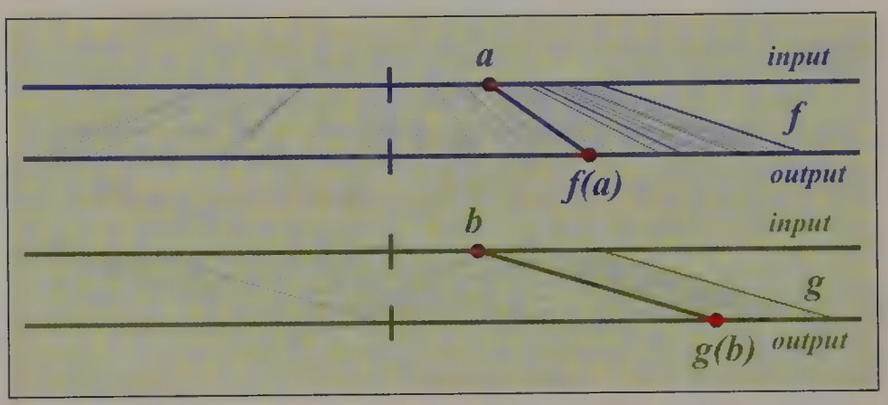


Fig. 4 Two dynagraphs are ready to be merged.

Similarly, students translate point y by 3 cm at 0° and label the dependent point appropriately. By tracing both points and dragging independent variable y (see **fig. 1b**), students create a visual representation of the behavior of this function.

Once students have constructed, manipulated, and observed the behavior of both functions, they are ready to investigate the composition.

As a first step, they erase the traces and again drag point x to form a visual image of the behavior of the dilation function. To explore what happens when point y follows the path of point $D_{C,2}(x)$, students position point y at the start of the trace of $D_{C,2}(x)$ (see **fig. 1c**) and drag y along the trace of $D_{C,2}(x)$ (see **fig. 1d**). The kinesthetic experience of dragging the independent variable of the second function along the path of the dependent variable of the first function helps tie the concept of composition to the concrete reality of the behavior of the two functions.

Next, students connect the two functions in a more permanent way: They merge point y to point $D_{C,2}(x)$, so that they become a single point, and edit the label of $T_{3\text{cm},0°}(y)$ by replacing y with $D_{C,2}(x)$. As students vary x once more (see **fig. 1e**), the behavior of the points and the resulting traces make clear the role of the three variables: x, $D_{C,2}(x)$, and $T_{3\text{cm},0°}(D_{C,2}(x))$.

This activity allows interesting extensions, such as the construction shown in **figure 2**, in which the student has repeatedly applied a composition of rotation and dilation to a picture of a leopard.

FUNCTION NOTATION

Students benefit from learning to use function notation early to denote geometric transformations (Usiskin et al. 1997). If the notation describes the functional relationship, it will make more sense to students. In the examples above, the class has agreed to use $D_{C,2}$ to denote dilation about point C by a scale factor of 2 and to use $T_{3\text{cm},0°}$ to denote translation by 3 cm at a direction of 0°. With these agreed-on designations, the relabeling in **figure 1e** makes sense to students, and they can read $T_{3\text{cm},0°}(D_{C,2}(x))$ as "the translation, by 3 cm at 0°, of the dilation, about C by a scale factor 2, of x."

The use of similar notation in various representations helps students master the notation itself and encourages students to recognize common elements that they can abstract from the different representations. Further, the descriptive role that the notation plays in geometric functions can help prevent the confusion that sometimes develops between $f(g(x))$ and $g(f(x))$ in algebraic functions.

DYNAGRAPH REPRESENTATION

The second representation of function composition uses dynagraphs (Goldenberg, Lewis, and O'Keefe 1992) to help students view and describe functions

as processes. As with geometric functions, dynagraph variables are points, and students can drag the independent variable continuously to change its value. In contrast with geometric functions, a dynagraph's variable points are restricted to axes that can be turned into number lines, creating the possibility of associating each position of a variable with a number. In this way, dynagraphs help students make the transition from the geometric to the numeric realm. **Figure 3a** shows a dynagraph with a segment connecting the independent and dependent variables; in **figure 3b**, the student has traced the connecting segment and dragged point d, creating the dynagraph equivalent of a table of values.

In the dynagraph activity, students use a prepared sketch containing two dynagraph functions. First, students manipulate, observe, and describe each function separately. They are encouraged to couch their descriptions in terms of the relative locations and speeds of the independent and dependent variables and are initially discouraged from interpreting the locations in numeric terms. For instance, "$j(d)$ moves in the same direction as d" is an appropriate observation, but "$j(d)$ increases when d increases" is not. Students record their observations for future reference.

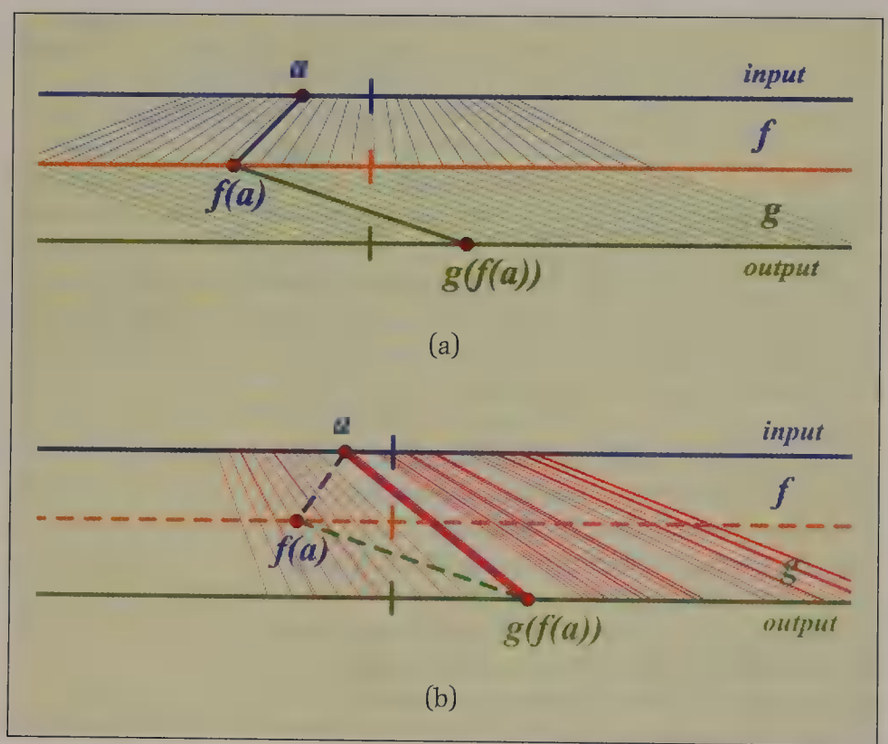


Fig. 5 Students merge the independent variable of function g with the dependent variable of function f and drag to observe traces (**a**) before directly connecting the input and output of the composed function (**b**).

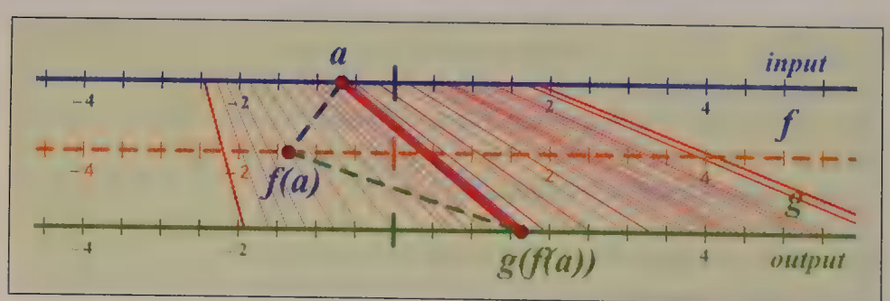


Fig. 6 Displaying coordinates on the axes creates a numeric representation of geometric variables.

Figure 4 shows two such dynagraphs along with the traces that indicate how the student has dragged the independent variables. After manipulating and describing both functions, students compose them as they did with geometric functions—by merging the independent variable of the second function with the dependent variable of the first. Students then relabel the points using function notation and describe the behavior of the resulting composed function. **Figure 5a** shows the composition, with the original segments traced to record the linked behavior of the two original functions; **figure 5b** shows the same composition but with a traced red segment used to show only the behavior of the composed function.

DYNAGRAPHS WITH NUMBERS

In **figure 6**, numbers have been added to the axes, allowing students to translate their geometric observations into numeric ones. For each previously recorded geometric observation, students are encouraged to formulate the equivalent—or even more specific—numeric observation, such as "$f(a)$ moves twice as fast as a" or "$g(b)$ is always three greater than b."

In the class discussion at the end of this dynagraph activity, it is important to ask students to discuss the similarities and differences in composition between geometric functions and dynagraph functions. This discussion is a valuable step in encouraging students to generalize, to move their thinking in the direction of an abstract concept of function composition that can be manifested in both geometric and numeric realms. During this discussion, probing questions can help students realize that a geometric function that dilates by a factor of 2 (the function labeled $D_{C,2}$) corresponds to the numeric function that multiplies by 2 (function f) and that a geometric function that translates to the right by 3 cm ($T_{3\text{cm},0°}$) corresponds to the numeric function that adds 3 (function g). This understanding—that multiplication and addition of numbers are the equivalents of dilation and translation of points—represents a deep connection between algebra and geometry and is an exciting and useful realization for students.

NUMERIC REPRESENTATION

The third representation of function composition, already foreshadowed by dynagraphs, involves calculating both functions numerically and then calculating the composed function. Students do this activity in two stages: They calculate a table of

$a = \boxed{3.00}$

$2 \cdot a = 6.00$

$(2 \cdot a) + 3 = 9.00$

a	$2 \cdot a$	$(2 \cdot a) + 3$
0.00	0.00	3.00
0.50	1.00	4.00
1.00	2.00	5.00
1.50	3.00	6.00
2.00	4.00	7.00
2.50	5.00	8.00
3.00	6.00	9.00

Fig. 7 Independent, intermediate, and dependent variables can be recorded in a table.

values, and then they plot the values on three horizontal axes, producing a dynagraph representation from the numeric one.

The student whose work is shown in **figure 7** has created an independent variable a; calculated the value of the first function, $2a$; used the result to calculate the value of the second function, $2a + 3$; and recorded in a table the results for several different values of the independent variable.

Next, the student constructs three horizontal lines with their determining points aligned, plots the three values, and animates the independent variable while tracing the connecting segments. The similarity of the result (see **fig. 8**) to that in **figure 5a** emphasizes the correspondence between numeric variables and points on a line. In **figure 5a**, the student drags a point on a number line and can interpret its position as a number; in **figure 8**, the student changes the value of a number and can interpret its value as a position on a line.

SYMBOLIC REPRESENTATION

The symbolic representation of function composition parallels and extends the numeric representation by formulating both functions in symbolic form and using the symbolic form to calculate the composed function and graph it on a Cartesian coordinate system.

The student whose work is shown in **figure 9a** has created the two given functions $f(x)$ and $g(x)$, evaluated them for arbitrary values of variables a and b, and then adjusted the value of b to match the value of $f(a)$. As shown in **figure 9b**, she has used GSP's **Function Calculator** to define the function $h(x) = g(f(x))$ and has evaluated $h(a)$ to confirm that it has the correct value. By using the composed

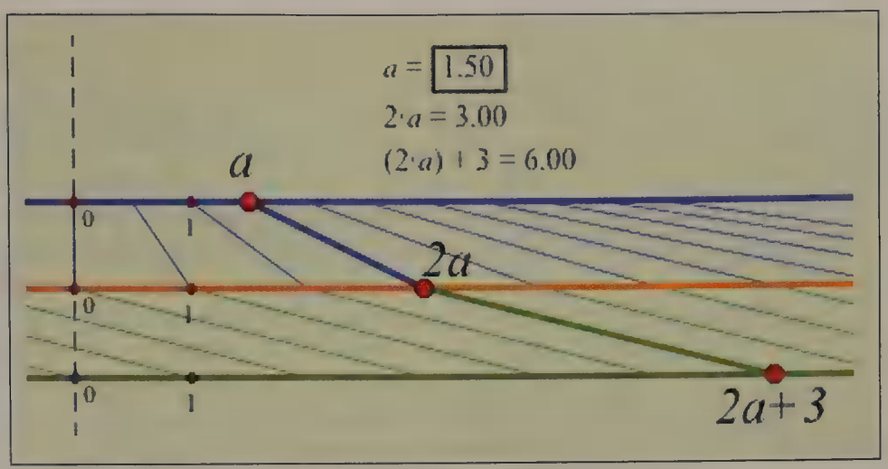


Fig. 8 When plotting values on parallel lines, students create a geometric representation of numeric variables.

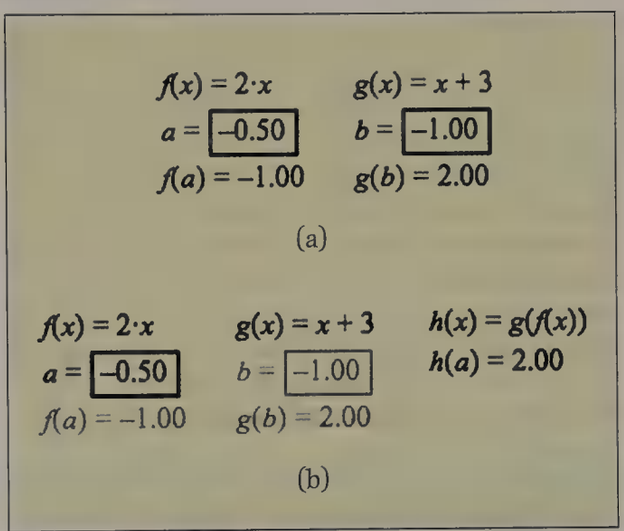


Fig. 9 Students use the output of f as input for g (**a**) and automate the process by defining a composed function h (**b**).

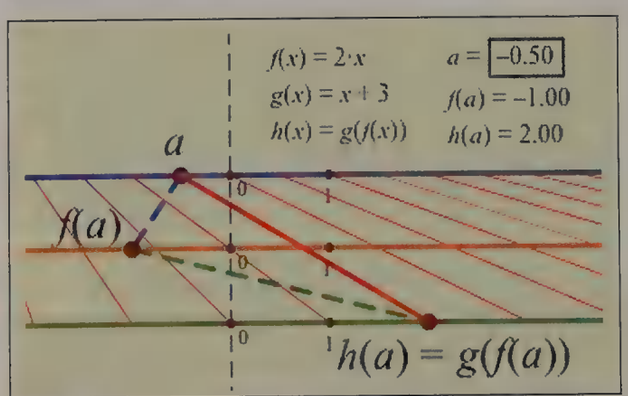


Fig. 10 Students use symbolic forms of functions f, g, and h; calculate numeric values; and plot the values to create a geometric representation.

function, the student no longer needs to adjust an intermediate variable manually; the grayed-out values in the illustration are now unnecessary.

As with the numeric representation, students should connect this representation to dynagraphs by creating an independent variable and using the symbolically defined functions to calculate the intermediate and final output variables. The result

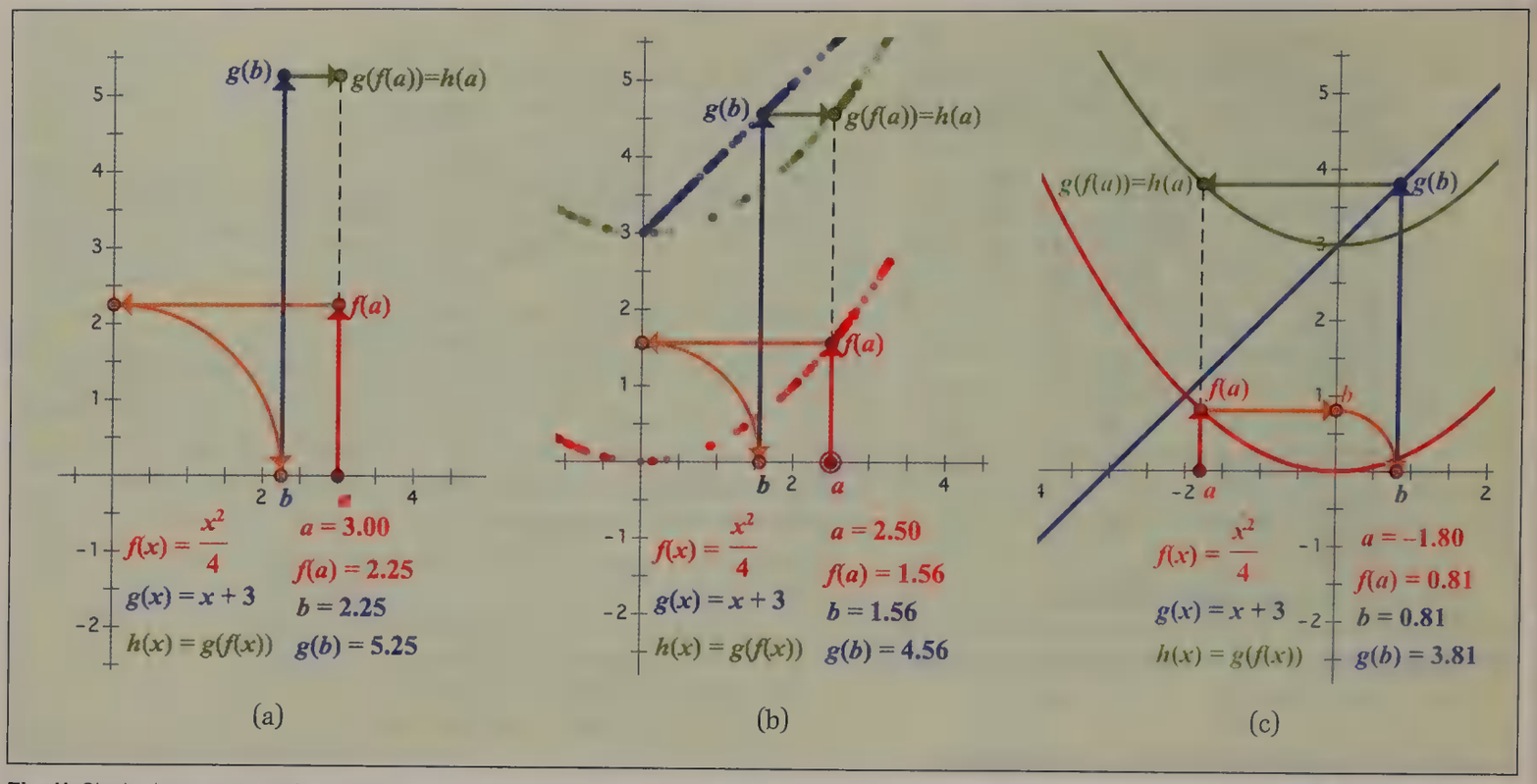


Fig. 11 Students use arrows to represent composition in Cartesian coordinates (**a**); trace all three output variables (**b**); and plot all three functions, showing their relationship to the arrows (**c**).

of plotting and animating these variables on parallel axes is shown in **figure 10**. The appearance is similar to the appearance of **figure 8**, but with explicit use of function notation.

CARTESIAN REPRESENTATION

Representing composition on Cartesian axes can be confusing to students for two reasons. First, the output (on the y-axis) of the first function serves as input (on the x-axis) to the second function, and this transfer of values from one axis to the other is unfamiliar and nonintuitive. Second, three different functions have to be graphed: the first, the second, and the composition. Whether they are graphed on the same or on different axes, there is no straightforward way to portray the relationships among them.

Because Cartesian graphs are a critical representation for so many purposes, students must form a mental image of how the composition is related to the Cartesian graphs of the three functions. The student whose work is shown in **figure 11a** has used a prepared sketch to animate four steps:

1. The red arrow locates the point with coordinates $(a, f(a))$.
2. An orange arrow transfers the value of $f(a)$ first to the y-axis and then to the x-axis and labels the new point b. Remember that $b = f(a)$.
3. The blue arrow locates the point with coordinates $(b, g(b))$—the same point as $(f(a), g(f(a)))$.
4. The green arrow arrives at the point $(a, h(a))$—the same point as $(a, g(f(a)))$.

The student whose work is seen in **figure 11b** has dragged point a along the x-axis while tracing the points representing the values of the three functions, in the process confirming that the traces correctly form the graphs of the three functions under consideration. As shown in **figure 11c**, she has displayed the graphs of the functions and has again dragged point a while observing how the changing values of the variables correspond to the arrows and to the shapes of the graphs.

Students finish this activity by editing the functions to explore the composition of a different pair of functions and then engaging in a final class discussion that concentrates on the similarities and differences among all the various representations: geometric, dynagraph, numeric, symbolic, and Cartesian.

In a discussion at the end of this final activity, teachers should once again ask students to discuss the similarities and differences between composition of functions using geometric, dynagraph, numeric, symbolic, and Cartesian representations. This discussion can focus on two aspects. First, students should use the similarities and differences to formulate a general definition of *function composition* by incorporating core features (the similarities) into the definition and excluding irrelevant details (the differences) from the definition. Second, students should describe how using function notation expresses the way in which the two functions are combined and how this notation can be incorporated into the definition.

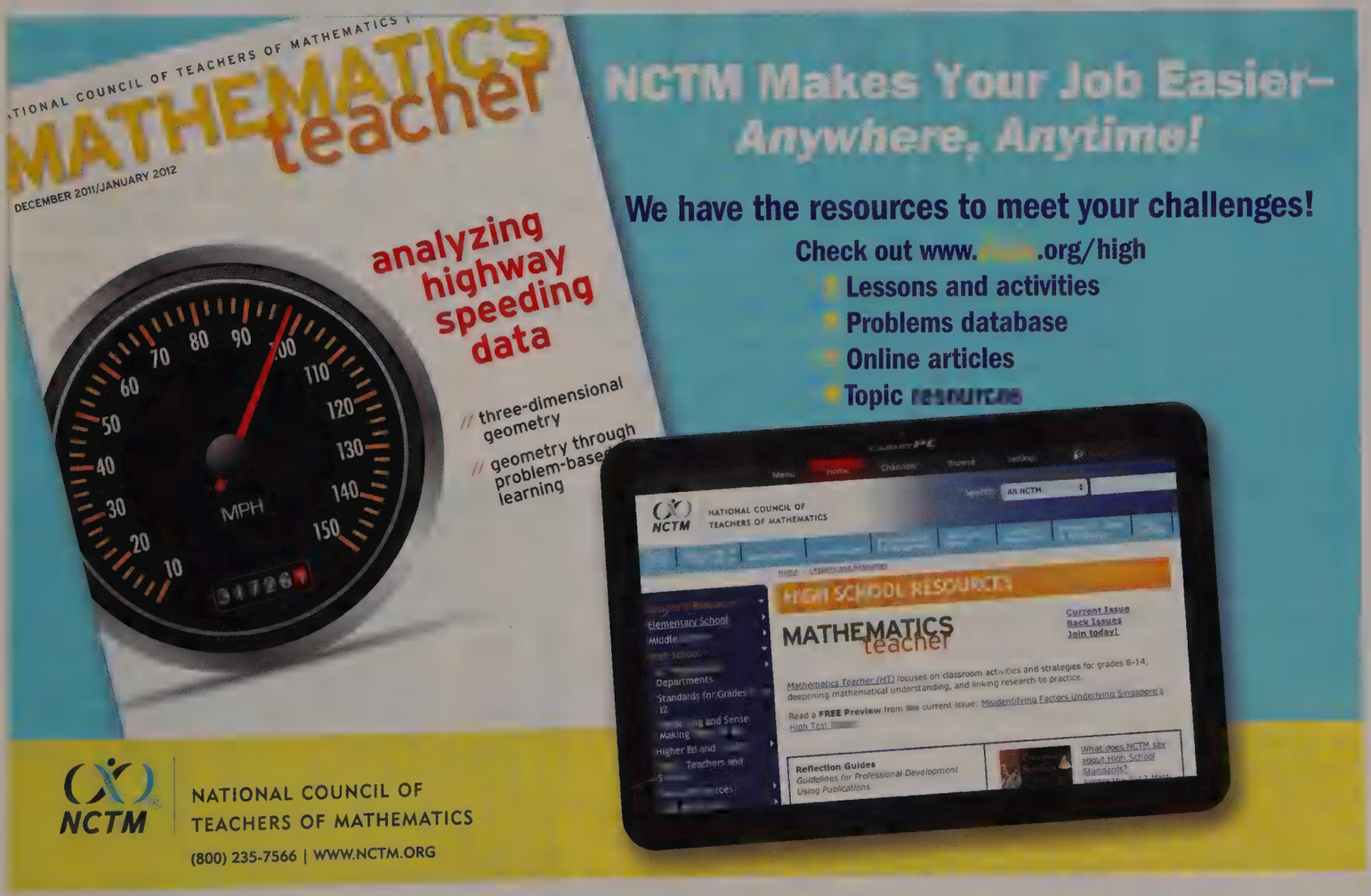
CONCLUSION

Each representation gives a different insight into functions and variables generally and into the concept of function composition specifically:

- Geometric functions allow students to construct the composition from scratch and to directly manipulate the independent variable while observing the behavior of both component functions.
- Dynagraphs allow students to vary the independent variable, modify the construction to create the composition, and ease the transition from geometric to numeric.
- Numeric functions concentrate on the input and output values directly as numbers, providing students a concrete experience in calculating composed numeric values.
- Symbolically expressed functions generalize numeric representations, leading students to develop abstract ways of thinking about composition.
- The Cartesian representation of composition clarifies the relationship between composition and Cartesian graphs, helping students overcome the difficulties of visualizing composition of such graphs.

In these activities, connections among representations have been emphasized by composing the same functions in both geometric and numeric realms. The first function scales (or multiplies) by 2, and the second translates by (or adds) 3. Scaling and translating are particularly important functions: Students use them to transform (compose) a parent function when they learn about function transformations (Cooney, Beckmann, and Lloyd 2010, Essential Understanding 4c, pp. 75–75).

By giving students the opportunity to create, manipulate, observe, discuss, and reflect on these five representations and to make connections among them, we enable students to develop and refine their concept of function composition. These memorable visual representations can aid understanding and recall and can help students generalize the concept of composition, overcome the tendency to tie function concepts to a single representation, and develop a more robust and mathematically rigorous understanding of the topic. The resulting deep understanding of composition will ease students' study of transformations of functions and inverse functions (which are simply compositions that produce the identity function). Finally, students' familiarity with geometric functions will support their later study of complex functions and functions of multiple variables.



WEB RESOURCES

The five activities described in this article are available for downloading:

- www.geometricfunctions.org/composition/multiplerepresentations/

Video suggestions for how to use these activities are available here:

- www.geometricfunctions.org/composition/multiplerepresentations/videos/

Additional activities using geometric functions to introduce functions and relations, domain, range, and function families are available here:

- www.geometricfunctions.org/activities/

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation Dynamic Number grant (grant no. 0918733). Opinions and views remain the authors' and not necessarily those of the National Science Foundation.

BIBLIOGRAPHY

Banchoff, Thomas. 2008. "Algebraic Thinking and Geometric Thinking." In *Algebra and Algebraic Thinking in School Mathematics,* 2008 Yearbook of the National Council of Teachers of Mathematics, edited by Carole Greenes and Rheta Rubenstein, pp. 99–110. Reston, VA: NCTM.

Chazan, Daniel. 2000. *Beyond Formulas in Mathematics and Teaching: Dynamics of the High School Algebra Classroom.* New York: Teachers College Press.

Chazan, Daniel, and Michal Yerushalmy. 2003. "On Appreciating the Cognitive Complexity of School Algebra: Research on Algebra Learning and Directions of Curricular Change." In *Research Companion to the NCTM Standards*, pp. 123–35. Reston, VA: National Council of Teachers of Mathematics.

Cooney, Thomas, Sybilla Beckmann, and Gwendolyn Lloyd. 2010. *Developing Essential Understanding of Functions, Grades 9–12.* Reston, VA: National Council of Teachers of Mathematics.

Goldenberg, E. Paul, Philip Lewis, and James O'Keefe. 1992. "Dynamic Representation and the Development of a Process Understanding of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy,* edited by Guershon Harel and Ed Dubinsky, pp. 235–60. *MAA Notes* no. 25. Washington, DC: Mathematical Association of America.

Hazzan, Orit, and E. Paul Goldenberg. 1996. "Students' Understanding of the Notion of Function." *Dynamic Geometry Environments: International Journal of Computers for Mathematical Learning* 6 (3): 263–91.

Kieran, Carolyn. 2006. "Research on the Learning and Teaching of Algebra." In *Handbook of Research on the Psychology of Mathematics Education (PME): Past, Present, and Future*, edited by Angel Gutiérrez and Paolo Boero, pp. 11–49. Rotterdam and Taipei: Sense Publishers.

Oehrtman, Michael C., Marilyn P. Carlson, and Patrick W. Thompson. 2008. "Foundational Reasoning Abilities That Promote Coherence in Students' Understandings of Function." In *Making the Connection: Research and Practice in Undergraduate Mathematics*, edited by Marilyn P. Carlson and Chris Rasmussen, pp. 27–42. Washington, DC: Mathematical Association of America.

Schoenfeld, Alan H., and Abraham Arcavi. 1988. "On the Meaning of Variable." *Mathematics Teacher* 81 (6): 420–27.

Selden, Annie, and John Selden. 1992. "Research Perspective on Conceptions of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy,* edited by Guershon Harel and Ed Dubinsky, pp. 1–16. *MAA Notes* no. 25. Washington, DC: Mathematical Association of America.

Steketee, Scott, and Daniel Scher. 2011. "A Geometric Path to the Concept of Function." *Mathematics Teaching in the Middle School* 17 (1): 48–55.

Usiskin, Zalman. 1999. "Conceptions of School Algebra and Uses of Variables." In *Algebraic Thinking, Grades K–12: Readings from NCTM's School-Based Journals and Other Publications,* edited by Barbara Moses, pp. 7–13. Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Zalman, et al. 1997. *UCSMP Geometry*. Glenview, IL: Scott Foresman.

Yerushalmy, Michal, and Beba Shternberg. 2001. "Charting a Visual Course to the Concept of Function." In *The Roles of Representation in School Mathematics*, 2001 Yearbook of the National Council of Teachers of Mathematics (NCTM), edited by Albert A. Cuoco, pp. 251–68. Reston, VA: NCTM.

SCOTT STEKETEE, steketee@kcptech.com, who taught mathematics for eighteen years in the Philadelphia public schools, develops The Geometer's Sketchpad curriculum and resources at KCP Technologies and is an adjunct instructor of mathematics methods at the University of Pennsylvania. DANIEL SCHER, dscher@kcptech.com, a senior scientist at KCP Technologies, is the principal investigator of the Dynamic Number project, funded by the National Science Foundation.

Thinking Like a Mathematician

Michael K. Weiss and Deborah Moore-Russo

The moves that mathematicians use to generate new questions can also be used by teachers and students to tie content together and spur exploration.

What does it mean to think like a mathematician? One of the great paradoxes of mathematics education is that, although we mathematics teachers are immersed in mathematical work every day of our professional lives, most of us nevertheless have little experience with the kind of work that research mathematicians do. Our ideas of what doing mathematics looks like are based mainly on our own experiences as students. Creating opportunities for students to engage in the kind of flexible thinking that is characteristic of mathematicians' practice can be a challenge for all of us.

NUMBEOS/ISTOCKPHOTO

Research has shown that most teachers, long before we ever enroll in our first education courses, already have firmly established ideas about teaching in general and mathematics instruction in particular; moreover, the ideas that we bring with us into teacher education programs often form the foundation on which we eventually build our own teaching (Millsaps 2000; Skott 2001). Before we entered the classroom as professionals, we were students for years—what Lortie (1975) referred to as an "apprenticeship of observation." The mathematics courses we took in high school as well as in our teacher education programs form some of our last preprofessional encounters with mathematics. Unfortunately, experience suggests that, for many of us, knowing mathematics is synonymous with being adept at manipulating symbols (Thompson 1992) and following memorized, step-by-step procedures (Ball 1988). Hence, many teachers consider mathematics "a discipline with a priori rules and procedures that … students have to learn by rote" (Handal 2003, p. 54).

Because of our own experiences, we often do not expect our students "to develop mathematical meanings" or "to use meanings in their thinking" (Thompson 2008, p. 45). It is, sadly, still unusual for students and teachers to be engaged in mathematical activities that require—or even allow for—flexible mathematical thinking. How often are students asked to pose their own problems or generalize a solution for a specific problem to a broader class of problems? In short, what do our students really learn about what it means to do mathematics?

Perhaps the most important questions are these: How can we break this self-perpetuating cycle? How can we, as teachers, encourage flexible mathematical thought if we ourselves had little opportunity to think flexibly when we were students?

HOW DO MATHEMATICIANS DO WHAT THEY DO?

One way to prepare ourselves and others to foster flexible mathematical thinking is to focus on the practices and habits of mind of research mathematicians—those who strive to generate new and refine existing mathematical ideas and methods. What does it mean to act like a mathematician?

In seeking to answer this question, Weiss (2009) analyzed a collection of narratives written by and about research mathematicians. This analysis reveals the fundamentally generative nature of engaging in mathematics, in which problem posing (asking fruitful and difficult questions of oneself and others) plays a role just as important as problem solving. The key role of problem posing in mathematics instruction has long been recognized. Silver (1994) noted that problem posing is a prominent feature not only of mathematical activity; it also features heavily in inquiry-oriented instruction and can help create an environment in which students are more engaged.

For example, Brown and Walter's (2004) "what-if-not?" strategy is a relatively simple means of generating new problems and promoting mathematical curiosity. Rather than accept a problem as it reads, a teacher asks instead: "What if this problem were not restricted to its given conditions?"

Consider the following scenario from high school geometry: The teacher has demonstrated the construction of a perpendicular bisector of a segment using a compass and straightedge, and students have practiced the method. The teacher then uses "what-if-not?" by asking: "What if the left and right arcs were not drawn with equal compass openings? Would the result still be a perpendicular bisector?"

The "what-if-not?" strategy is not limited to exploring geometry. Suppose that in a unit on

It is, sadly, still unusual for students and teachers to be engaged in mathematical activities that require—or even allow for—flexible mathematical thinking.

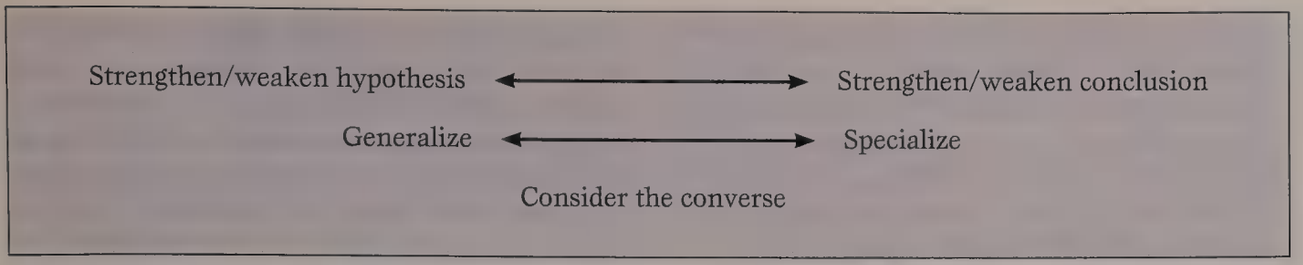


Fig. 1 Generating new problems from old ones can be accomplished using these strategies.

elementary statistics the teacher has provided the class with a set of three numbers (10, 20, and 30), and students have found that 20 is both the mean and the median of the set. Using "what-if-not?" the teacher could ask, "How would the mean and median change if the three numbers were not equally spaced?" More specifically, the teacher could ask, "How would the middle number have to be changed if we wanted the mean to be larger (or smaller) than the median?"

The "what-if-not?" strategy is one format for generating new questions. In a broader study of what research mathematicians care about and how they generate new problems from old ones, Weiss (2009) identified five "generative moves" for problem posing (see **fig. 1**). Here are some examples from geometry and algebra of how these moves can be used in the classroom and how common instructional activities can be changed slightly to include instances of authentic mathematical work.

Suppose that a class of geometry students has been studying the properties of triangles and has found that the three angle bisectors of any acute triangle always intersect in a single point. The following scenarios show how teachers can use the five generative moves to guide students in exploring the related mathematical terrain:

- The teacher might ask, "Does it really matter whether the triangle is acute or not?" Investigating this question could lead the class to conclude that the restriction to the acute case was unnecessary and that the conclusion holds for all triangles—a case of weakening the hypothesis, the first generative move shown in **figure 1**.
- The teacher might encourage students to strengthen the conclusion of what has been proven—for example, by offering, "Not only do they intersect at a single point, but that point is the center of a circle that can be inscribed in the triangle."
- The teacher might encourage students to generalize their findings—perhaps by asking, "What if we construct the angle bisectors of other polygons? Do they meet at a point? If not, what happens?"
- The teacher might ask students to specialize their findings—for example, by observing, "If you construct angle bisectors in an equilateral triangle, what else is true? For example, is the point of intersection equidistant from the vertices of the triangle?"
- If students observe this last property, they then might be encouraged to consider the converse question: "If the angle bisectors of a particular triangle meet at a point that is equidistant from the vertices of the triangle, does this result mean that the triangle in question must be equilateral?"

These examples illustrate how the five generative moves for problem posing can be used to describe and promote the practice of wondering mathematically about what is true, a core component of flexible mathematical thinking. More examples could be generated almost without limit by iterating and recombining these moves.

Again, the use of these generative moves is not limited to geometry. Suppose that a first-year algebra class is at the very beginning of its study of linear functions. The class has created a table of values listing eight pairs of numbers (x, y) that satisfy the equation $3x - 2y = 0$, perhaps through trial and error, by choosing values for one of the variables and then solving the resulting equation for the other, or through some combination of methods. Students have plotted these pairs of numbers on a graph and have noted that they all appear to lie on a straight line through the origin. What could happen next?

- Students might consider the converse question: Does every point on that line satisfy the same equation?
- Having come to believe that the answer to the question above is yes, students might then consider the converse again and ask: Does every point not on the line fail to satisfy the equation? (Although this statement is technically the inverse, not the converse, of the previous statement, the two are logically equivalent.)
- Once students have identified the points on the line as precisely the set of all pairs of values that satisfy the equation, students could attempt to strengthen the conclusion by saying more about the line—for example, that the line rises by 3 units for every horizontal change of 2 units.

- Students could continue by asking questions such as this one: How would the graph be changed if the right side of the equation were not zero but some other constant? Such questions are an application of the generalization move (and, it should be noted, a direct application of the "what-if-not?" template).

Although all algebra teachers address the same basic facts about linear functions, these examples are meant to emphasize how the generative moves can be used to tie these facts together into a cohesive set of inquiry-driven activities. Modeling and promoting flexible mathematical thinking for students does not necessarily involve learning new ideas. It can also play a role in helping us recast the ideas that we already teach, tying facts and procedures into a coherent narrative.

The generative moves can be used to tie facts together into a cohesive set of inquiry-driven activities.

STUDENTS AS RESEARCHERS

Many teachers include a focus on problem solving in their practice. Some of the ideas above were incorporated in the teaching of problem posing in a high school geometry class. As a year-end assessment, students were assigned a research project on the mathematical properties of *duals*—figures formed by joining the midpoints of adjacent sides of a polygon (see **fig. 2**). The instructions for the project stated, in part:

> This assignment is different from what you are probably accustomed to, because you have to *generate the questions*—not just answer them ... (*a*) to pose interesting questions; (*b*) to investigate your questions, by drawing lots of pictures, making measurements, and looking for patterns; (*c*) to formulate conjectures, and present evidence to support your conjectures; (*d*) when possible, to answer your questions, and *prove* (or *disprove*) your conjectures (at which point they stop being conjectures and become *theorems*).

Students were encouraged to report on their investigation of questions that they could not answer as well as those that they could.

Students produced a wide range of questions, conjectures, and results. Most proved that the dual of any triangle is a half-scale, 180°-rotation of the original triangle. Most also investigated the duals of special quadrilaterals. Many students did more than simply list a set of disconnected results; they used the generative moves to link their findings into a coherent theory.

One student began with an observation (and a proof) that the dual of a square is always a square. He then asked whether the converse is true: If the dual of a quadrilateral is square, must the "outside" quadrilateral be square as well? On finding a counterexample, he returned to his original finding and then weakened the hypothesis: What if the "outside" quadrilateral is not a square but just a rectangle or a rhombus? These questions led to the discovery that the dual of a rectangle is a rhombus (and vice versa) and, hence, that the *second dual* (the dual of the dual) of a rectangle or a rhombus is another figure of the same type. The student then strengthened the conclusion, proving that the second dual of a rectangle or a rhombus is actually similar to the original figure, with a scale factor of one-half. Finally, he weakened the hypothesis of this result, finding that any parallelogram is similar to its second dual.

Another student also found that the dual of a rectangle is a rhombus (and vice versa) but then went off in an entirely different direction. She wrote that while doodling in her notebook during her world history class (a fine example of mathematical wondering, although perhaps not one that her world history teacher would have approved of), she became curious about what it would mean to form the dual of a cube or another solid. Two ideas occurred to her: She could join the midpoint of each edge to the midpoints of the neighboring edges, or she could join the center of each face to the centers of the adjacent faces. Sketching both possibilities, she discovered that, using the second approach, the dual of a cube is an octahedron and the dual of an octahedron is

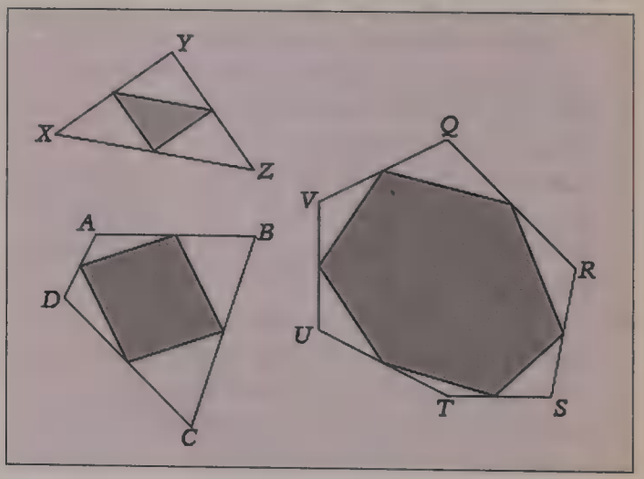


Fig. 2 Connecting consecutive midpoints of the sides of a polygon creates the figure's dual.

another cube—a result analogous to what she had already shown about rectangles and rhombi. She also found that the dual of a tetrahedron is an "upside-down" tetrahedron, analogous to what she already knew about triangles. On the basis of these findings, she decided that the second method is the "right" way to construct duals of solids—an impressive example of generalization as a generative move.

Not only did students demonstrate outstanding mathematical work; many of them took evident pride of ownership in what they had created. One wrote on the cover of her report, "The most FUN final ever!!!" Two others (separately) thanked the teacher when they turned in their work. In the years since teaching that class, the teacher has been surprised, on three separate occasions, by former students (now in college) reminiscing enthusiastically about the research projects. One student referred to it as "the first time I ever really got to *think* about math."

CONCLUSION

For our classrooms to become venues where students experience mathematical activities that foster flexible thinking, we need to change how we as teachers think about and do mathematics. Teachers at all levels should make the mechanisms of problem posing explicit and draw attention to how they can be used to navigate through open-ended problems. Through engagement in such mathematical activities, teachers and their students might come to view mathematics differently. By extension, these teachers might come to teach mathematics in a manner that is more authentic to actual mathematical practice.

REFERENCES

Ball, Deborah L. 1988. "Knowledge and Reasoning in Mathematical Pedagogy: Examining What Prospective Teachers Bring to Teacher Education." PhD diss., Michigan State University.

Brown, Stephen I., and Marion I. Walter. 2004. *The Art of Problem Posing*. 3rd ed. Hillsdale, NJ: Lawrence Erlbaum.

Handal, Boris. 2003. "Teachers' Mathematics Beliefs: A Review." *The Mathematics Educator* 13 (2): 47–57.

Lortie, Dan. 1975. *Schoolteacher: A Sociological Study*. Chicago: University of Chicago Press.

Millsaps, Gayle. 2000. "Secondary Mathematics Teachers' Mathematics Autobiographies: Definitions of Mathematics and Beliefs about Mathematics Instructional Practice." *Focus on Learning Problems in Mathematics* 22 (1): 45–67.

Silver, Edward. 1994. "On Mathematical Problem Posing." *For the Learning of Mathematics* 14 (1): 19–28.

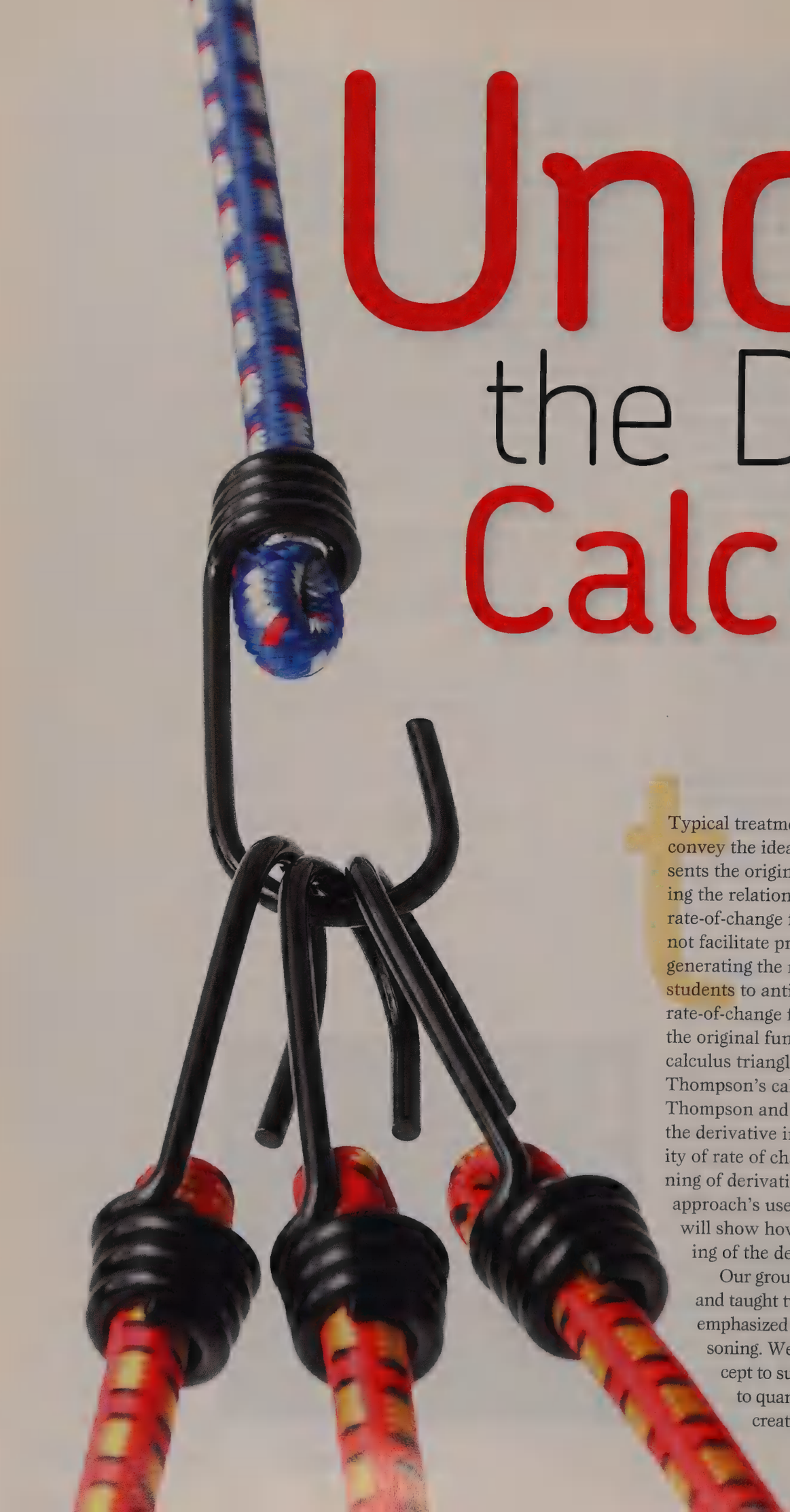
Skott, Jeppe. 2001. "The Emerging Practices of a Novice Teacher: The Roles of His School Mathematics Images." *Journal of Mathematics Teacher Education* 4 (1): 3–28.

Thompson, Alba G. 1992. "Teachers' Beliefs and Conceptions: A Synthesis of the Research." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Douglas A. Grouws, pp. 189–236. Hillsdale, NJ: Lawrence Erlbaum.

Thompson, Patrick W. 2008. "Conceptual Analysis of Mathematics Ideas: Some Spadework at the Foundation of Mathematics Education." In *Proceedings of the Joint Meeting of the International Group for the Psychology of Mathematics Education and the North American Group for the Psychology of Mathematics Education*, edited by Olimpia Figueras, José L. Cortina, Silvia Alatorre, Teresa Rojano, and Armando Sépulveda, vol. 1, pp. 45–64. Mexico: Cinvestav-UMSNH.

Weiss, Michael. 2009. "Mathematical Sense, Mathematical Sensibility: The Role of the Secondary Geometry Course in Teaching Students to Be Like Mathematicians." PhD diss., University of Michigan.

MICHAEL K. WEISS, mweiss@msu.edu, is assistant professor of mathematics education at Michigan State University in East Lansing. A former research mathematician and high school mathematics teacher, he is interested in how authentic mathematical practices can cultivate students' mathematical sensibility. DEBORAH MOORE-RUSSO, dam29@buffalo.edu, who taught high school mathematics for thirteen years, teaches in the graduate school of education at the University at Buffalo (SUNY). Her primary research interests include spatial visualization, communication, and reasoning.



Under
the Deriva
Calculus

Typical treatments of the derivative do not clearly convey the idea that the derivative function represents the original function's rate of change. Revealing the relationship between a function and its rate-of-change function for static values of x does not facilitate productive ways of thinking about generating the rate-of-change function or allow students to anticipate the graphical behavior of the rate-of-change function by examining a graph of the original function. Accordingly, we propose the calculus triangle approach. This approach builds on Thompson's calculus research (Thompson 1994; Thompson and Silverman 2008), which introduces the derivative in a way that maintains the centrality of rate of change as a conceptual underpinning of derivative. By providing examples of the approach's use in novel and routine settings, we will show how it facilitates a mature understanding of the derivative.

Our group of teacher-researchers had designed and taught two university calculus courses that emphasized rate of change and quantitative reasoning. We proposed the calculus triangle concept to support students in attending explicitly to quantities and constructing a method for creating and tracking the ratio of changes

standing ve through the Triangle

We know derivatives are about rates. Why don't our students?

Eric Weber, Michael Tallman, Cameron Byerley, and Patrick W. Thompson

in quantities to produce a rate-of-change function. Further, the calculus triangle approach allows students to reason flexibly across mathematical domains, such as differentiation and accumulation, as well as across graphical representations.

Historically, the derivative was constructed as a way to represent and measure the rate at which one quantity changes with respect to another quantity. However, many students are taught in a way that enables them to solve calculus problems without attending to rates of change. Carlson and colleagues (2002) found that second-semester calculus students were unable to produce a qualitative graph that expressed the height of water in a bottle as a function of the water's volume. These students used memorized properties of second derivatives but could not relate inflection points in the graph of the function to changes in the bottle's width.

In addition, a number of researchers reported students' difficulties in creating graphical representations of a function's rate-of-change function (Tall 1986; Ubuz 2007). These researchers found that students often focused on computing derivatives without connecting the derivatives they computed and evaluated to a function's rate of change at specific points in its domain (Ubuz 2007).

THE CENTRALITY OF RATE OF CHANGE

Calculus textbooks' attention to velocity and development of the limit definition of the derivative suggest that textbook authors intend to develop the idea of derivative as a rate of change. In our efforts to understand the disconnect between what textbooks present and what students understand, we have focused heavily on how students think about derivatives. We surveyed the best-selling calculus books in the United States and found that they qualitatively generated the derivative function in different ways.

In most textbooks, the secant does not slide through the function's domain. Rather, one intersection of the secant line slides toward the other intersection, creating successively better approximations of the tangent line. The tangent line, however, slides through the function's domain. Ferrini-Mundy and Graham (1991) found that students often struggled to envision and make sense of a sliding secant line and its relationship to rate of change on a small interval and believed that the secant line collapsed into a single point to create the sliding tangent. Our observations suggest that this problem persists today.

Even if students were successful in envisioning the slope of the tangent line as representing the rate

triangle approach from the traditional approach is that in the calculus triangle approach, generating the derivative function *precedes* letting h approach zero; in the traditional approach, the limiting process comes first.

When focusing on the calculus triangle, students are better suited to think about infinitesimal rate of change near a point. This approach contrasts with thinking about rate of change at a point, where quantities are not changing; then, discussing rate of change becomes problematic. A classroom discussion about the relative accuracy of the approximation for sufficiently small values of h can be framed so that students' reasoning in relation to generating approximations of the actual derivative functions can be isomorphic to more formal definitions of uniform convergence.

IMPLICATIONS AND CONCLUSION

The traditional approach to developing the derivative function can accomplish one of the following two goals but not both: an understanding that derivative functions are fundamentally about rates of change; or an understanding that the derivative function can be generated by developing a ratio of the changes in the output quantity measured in units of the input quantity through the domain of the function f. The calculus triangle approach supports students in establishing connections between average rate of change and the derivative function. At the same time, it does not compromise the potential for the derivative function to be generated by explicitly measuring the rate of change of $f(x)$ with respect to x.

Traditional treatments of calculus use the definition of *derivative* in an axiomatic approach to develop further rules of differentiation. This process emphasizes the derivative as an operator on functions instead of the centrality of rate of change. The calculus triangle approach allows students to understand the derivative function as tracking the ratio between changes in two quantities. If students are able to attend to the derivative as a rate of change, this ability supports them in understanding the process of differentiation as an operator that measures a rate of change.

Courses in single and multivariable calculus and differential equations require students to perform symbolic computations as well as interpret the meaning of those computations. The calculus triangle approach supports students in developing both computational fluency and in interpreting the results of those computations meaningfully. As students approach novel problem-solving situations, they are equipped with ways of thinking, particularly about the derivative function as a rate of change, that are necessary to reason through these situations. By thinking about the meaning of the computations, students are able to draw connections between major ideas.

REFERENCES

Carlson, Marilyn P., Sally Jacobs, Edward Coe, Sean Larsen, and Eric Hsu. 2002. "Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study." *Journal for Research in Mathematics Education* 33 (5): 352–78.

Ferrini-Mundy, Joan, and Karen G. Graham. 1991. "An Overview of the Calculus Curriculum Reform Effort: Issues for Learning, Teaching, and Curriculum Development." *American Mathematical Monthly* 98 (7): 627–35.

Tall, David. 1986. "Building and Testing a Cognitive Approach to the Calculus Using Interactive Computer Graphics." PhD diss., University of Warwick.

Thompson, Patrick W. 1994. "Images of Rate and Operational Understanding of the Fundamental Theorem of Calculus." *Educational Studies in Mathematics* 26 (2–3): 229–74.

Thompson, Patrick W., and Jason Silverman. 2008. "The Concept of Accumulation in Calculus." In *Making the Connection: Research and Teaching in Undergraduate Mathematics,* edited by Marilyn P. Carlson and Chris Rasmussen, pp. 43–52. Washington, DC: Mathematical Association of America.

Ubuz, Behiye. 2007. "Interpreting a Graph and Constructing Its Derivative Graph: Stability and Change in Students' Conceptions." *International Journal of Mathematical Education in Science and Technology* 38 (5): 609–37.

ERIC WEBER, Eric.Weber@oregonstate.edu, is assistant professor of mathematics education at Oregon State University in Corvallis. He is interested in how technology can support visualizing advanced mathematical ideas. MICHAEL TALLMAN, michael.tallman@asu.edu, and CAMERON BYERLEY, cameron.byerley@asu.edu, are PhD candidates in mathematics education at Arizona State University in Tempe. Tallman is interested in characterizing mathematical knowledge for teaching secondary school mathematics. Byerley researches how students think about calculus and supports Teach for America teachers. PATRICK W. THOMPSON, pat.thompson@asu.edu, is professor of mathematics education in the School of Mathematical and Statistical Sciences at Arizona State University. He teaches mathematics education courses as well as research courses for doctoral students.

THE NATION'S **PREMIERE** MATH EDUCATION EVENT

NCTM 2013 Annual Meeting & Exposition

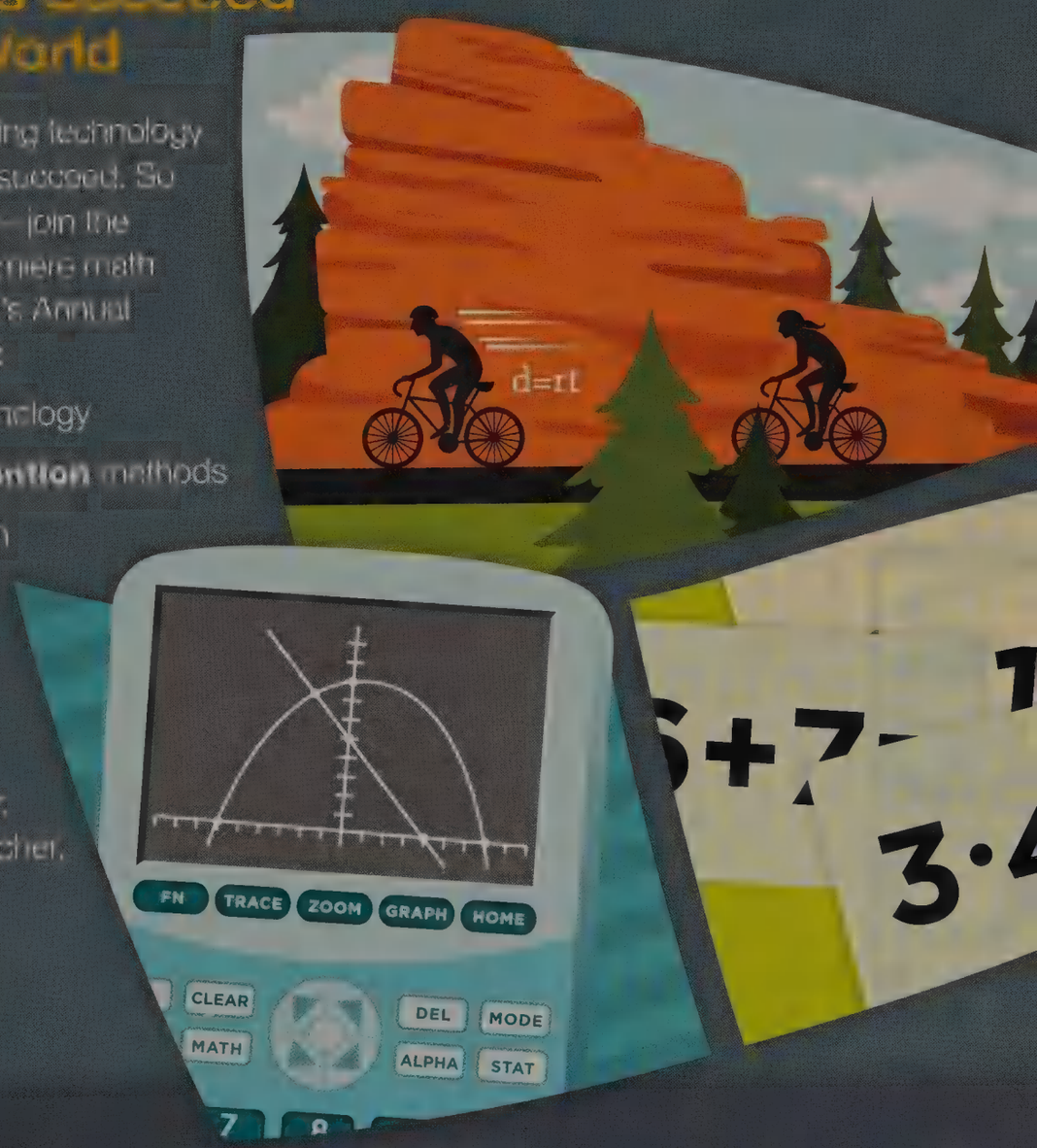
APRIL 17-20 • DENVER, COLORADO

Help Your Students Succeed in a Competitive World

In a global society with rapidly changing technology your students need the right tools to succeed. So take the next step to help them grow – join the best and brightest at the nation's premiere math education event. By attending NCTM's Annual Meeting you and your colleagues will:

- Discover and test the latest technology
- Learn new and effective **intervention** methods
- Hear about **essential** education topics from the experts
- Refine your **assessment** techniques
- And more!

Whether you're a classroom teacher, coach, administrator, preservice teacher, or math specialist, NCTM's Annual Meeting has something for you.



NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Visit **www.nctm.org/meetings** for

DECEMBER

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Fostering Flexible Mathematical Thinking

Flexible mathematical thinking—the ability to generate and connect various representations of concepts—is useful in understanding mathematical structure and variation in problem solving. Each of this month's problems can be solved through multiple approaches.

Solve the following system of equations:

$$x + y = 11$$
$$3x - y = 5$$

4

The coordinates of the endpoints of a diameter of a circle are (–2, 7) and (6, 11). Write an equation of the circle.

1

Solve the following system of equations:

$$x + y + z = 4$$
$$x - 2y - z = 1$$
$$2x - y - 2z = -1$$

5

Find the number of seats in a theater containing 30 rows if the first row has 14 seats and each row after that has 2 more seats than the previous row.

2

Four cards are drawn simultaneously from a standard deck of 52 cards. Find the probability that all four are picture cards (jack, queen, or king).

6

Write the equation of a parabola in the form $y = ax^2 + bx + c$ where a, b, and c are integers with no common factors, $a > 0$, and the zeroes of the parabola are –5/3 and 7/2.

3

Ralph has 10 identical pennies and 4 pockets in his jeans. In how many different ways can he store the pennies in his pants pockets? Putting all 10 in each pocket is considered a different arrangement.

7

The value of a new car typically decreases by 35% the first year and, on average, by 60% over the first five years. If the rate of decrease is constant after the first year, determine the value of that constant.

8

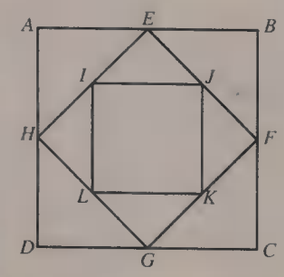
Divide a line segment of given length into five congruent segments using only a straightedge and compass.

9

In the three-digit number $95d$, d is the units digit. If the digits are reversed and the resulting number is subtracted from the original number, the result will consist of the same three digits in a different order. Find d.

10

$ABCD$ is a square, and E, F, G, H, I, J, K, and L are midpoints of their respective line segments. Determine the area of $IJKL$ if the area of $ABCD$ is 12.



11

The equations of two circles are

$$x^2 + y^2 = 16$$

and

$$(x - 5)^2 + y^2 = 9.$$

12

Three bags look identical, but their contents differ. One bag contains 2 pennies, the second contains 2 nickels, and the third contains 1 penny and 1 nickel. A bag is chosen at random, and one coin is removed. If the coin is a penny, what is the probability that the remaining coin in the bag is also a penny?

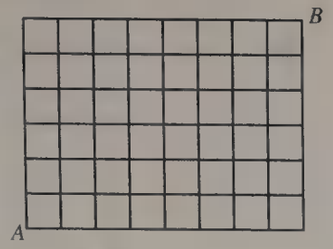
13

Trapezoid $ABCD$ with bases AD and BC has diagonals meeting at E. The line through E parallel to the bases intersects the nonparallel sides at F and G. If $BC = 10$ and $AD = 15$, compute FG.

14

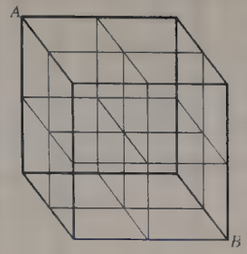
Alf and Bob cross a lake with a single-seat kayak. They can paddle at 7 mph but swim at 2 mph. They start out together—one paddling, one swimming. Later, the kayaker anchors the craft and starts swimming. When the swimmer reaches the kayak, he gets in and starts paddling. If the men arrive at their destination at the same time

How many different paths from A to B are possible if the only allowable moves are up or to the right?



16

All the lines in the diagram represent vertical and horizontal girders in a partially completed building. In how many ways can a construction worker moving along the girders get from point A to point B if each step brings him closer to his goal?



17

Find the equation of a circle in the form

$$(x - h)^2 + (y - k)^2 = r^2$$

that contains the points (–5, 12), (9, 14), and (3, –4).

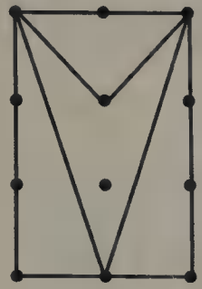
18

Compute a two-digit number that satisfies the following condition:

4 • the number – 5
= the number with digits reversed

19

Given the 12 lattice points, determine the ratio of the area of the arrowhead to the area of the rectangle.



20

Compute the sum:

$$\sin(\pi/7) + \sin(4\pi/7) + \sin(7\pi/7) + \sin(10\pi/7) + \sin(13\pi/7)$$

21

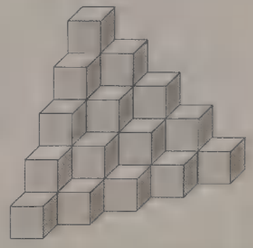
Two circles with radii 25 and 16 are tangent externally. The two common external tangents intersect the larger circle at A and B and the smaller circle at C and D. The common internal tangent intersects the external tangents at E and F. Find EF.

22

A pair of opposite sides of a square whose lengths are s units are each subdivided into n congruent parts. The corresponding points on the two sides are then joined to form n congruent rectangles, each with perimeter p. Find the area of the square in terms of n and p.

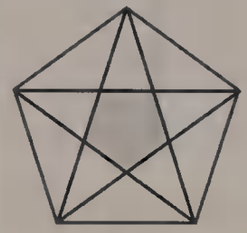
23

The tower is made up of five layers of solid cubes with no gaps. How many individual cubes are in the tower?



24

Determine the ratio of the area of the outer regular pentagon to the area of the shaded inner regular pentagon.



25

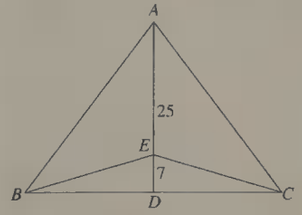
A line l intersects the positive x- and y-axes at A $(a, 0)$ and B $(0, b)$, respectively. Lines parallel to the x- and y-axes are drawn through a point Q on l, intersecting the x-axis at R and the y-axis at P. O is the origin, and rectangle $OPQR$ and triangle QAR have the same area. Find the area of triangle BPQ in terms of a and b.

26

A triangle is inscribed in a circle. The ratio of the measures of the angles is 5 : 12 : 13. The shortest side of the triangle is a side of a regular n-sided polygon inscribed in the same circle. Determine n.

27

Triangle ABC is isosceles with $AB = AC$. Determine the perimeter of $\triangle ABC$ if $BD = DC$, $AE = 25$, $ED = 7$, and $m\angle BEC = 2 \cdot m\angle BAC$.



28

A rectangle has sides that measure 5 and 12. Each side is divided into 7 congruent segments. Draw one of the diagonals of the rectangle and the 12 lines parallel to the diagonal that connect the corresponding points of the subdivided sides. Find the sum of the lengths of all 13 diagonal segments.

29

The arithmetic sequence $a_1, a_2, a_3, \ldots, a_n$ has a common difference of 5, and the series $a_1 + a_2 + a_3 + \cdots + a_{80} = 2012$. Determine the value of

$$a_2 + a_4 + a_6 + \cdots + a_{80}.$$

30

A circle and a rectangle have the same center. The radius of the circle is 4, and the sides of the rectangle are 6 and 8. Determine, to the nearest hundredth, the positive difference between the area inside the circle but outside the rectangle and the area inside the rectangle but outside the circle.

31

© National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 20191-1502

SOLUTIONS to calendar

Looking for more Calendar problems?
Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher.*

Edited by ***Margaret Coffey,*** *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and* ***Art Kalish,*** *artkalish@verizon.net, Syosset High School (retired), Syosset, NY 11791*

Problems 18, 21, 23, and 24, by Rich Kalman, are from the New York City Interscholastic Math League. Problem 20 was suggested by William Jamski of Indiana University Southeast in New Albany, IN.

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Other sources of problems in calendar form available from NCTM include *Calendar Problems from the "Mathematics Teacher"* (a book featuring more than 400 problems, organized by topic; stock number 12509, $22.95) and the 100 Problem Poster (stock number 13207, $9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.—Eds.

1. $(x-2)^2+(y-9)^2=20$. Find the center of the circle by determining the midpoint of the diameter by averaging the x- and y-coordinates, respectively, to get (2, 9). Then apply the distance formula using the center of the circle and one of the endpoints to find the length of the radius:

$$r=\sqrt{(6-2)^2+(11-9)^2}=\sqrt{20}$$

Therefore, the equation of the circle is $(x-2)^2+(y-9)^2=20$.

Alternate solution: An equation of the circle is $(x+2)(x-6)+(y-7)(y-11)=0$. It is a worthwhile exercise to show that the two results are equivalent. Another worthwhile exercise is to prove the generalization that if the endpoints of the diameter of a circle are (x_1, y_1) and (x_2, y_2), then the equation of the circle is $(x-x_1)(x-x_2)+(y-y_1)\cdot(y-y_2)=0$.

2. 1290. Apply the formulas for arithmetic sequences to determine the number of seats in the last row. The formula for the nth term in an arithmetic sequence is $a_n=a_1+(n-1)d$, so $a_{30}=14+(30-1)\cdot 2=72$. The formula for the sum of n terms in an arithmetic sequence is $S_n=n(a_1+a_n)/2$, so $S_{30}=30(14+72)/2=1290$.

Alternate solution: The seats in the theater are arranged in the shape of a trapezoid. Therefore, the number of seats is equivalent to the area of the trapezoid measured in "seat" units. The formula for the area of a trapezoid is $A=h\cdot(b_1+b_2)/2=h(median)$. The median is the average of the bases, or the number between rows 15 and 16. Use $a_{15}=14+(15-1)\cdot 2=42$. Thus, the median = 43, and the number of seats is 30(43) = 1290.

3. $y=6x^2-11x-35$. Use the sum and the product of the roots to determine the equation: The sum is $-5/3+7/2=11/6$, and the product is $-5/3\cdot 7/2=-35/6$. Thus, an equation is $0=x^2-11x/6-35/6\rightarrow y=6x^2-11x-35$. All other equations with the same roots would just be multiples of the right side of this equation.

Alternate solution: The roots are $x=-5/3$ and $x=7/2$, so $3x=-5\rightarrow 3x+5=0$ and $2x=7\rightarrow 2x-7=0$. Finally, $(3x+5)(2x-7)=y$, so $6x^2-11x-35=y$.

4. (4, 7). Add the two equations together to get $4x=16\rightarrow x=4$. Substitute into either original equation to find $y=7$.

Alternate solution: Solve $3x-y=5$ for y, resulting in $y=3x-5$. Substitute this expression into the first equation to get $x+(3x-5)=11\rightarrow 4x=16\rightarrow x=4$, so $y=3(4)-5=7$.

5. (2, –1, 3). Subtract the second equation from the first to get $3y + 2z = 3$; subtract the third equation from twice the first equation to get $3y + 4z = 9$. The difference between these two results is $2z = 6$, so $z = 3$. Substituting back into the original equations yields $y = -1$ and $x = 2$.

Alternate solution 1: Solve the first equation for x and substitute the result into both the second and third equations. So $x = 4 - y - z$; therefore, $4 - y - z - 2y - z = 1 \rightarrow -3y - 2z = -3$ and $2(4 - y - z) - y - 2z = -1 \rightarrow -3y - 4z = -9$. These two results are equivalent to the two equations in the original solution, so solve as above.

Alternate solution 2: Use Cramer's rule to set up and evaluate appropriate determinants. This solution can also be completed using matrices and Gaussian elimination techniques.

6. 495/270725 = 99/54145 ≈ .0018. The sample space contains ${}_{52}C_4$ different selections of four cards, and there are ${}_{12}C_4$ selections that contain only picture cards. Thus, P(four picture cards) = 495/270725 = 99/54145 ≈ .0018.

Alternate solution: Multiply the probability of success for each of the four events. This approach results in 12/52 · 11/51 · 10/50 · 9/49 = 495/270725.

7. 286. Make an organized list of ordered 4-tuples and count. One way to organize the list is to place the numbers in numerical order and determine the number of equivalent arrangements of each 4-tuple. For example, (10, 0, 0, 0) can be arranged to form four 4-tuples. The next arrangement is (9, 1, 0, 0), and there are twelve ways to reorder these four numbers. The rest of the list is shown in the charts below.

4-tuple	8200	8110	7300	7210	7111	6400	6310	6220	6211	5500	5410
Rearrangements	12	12	12	24	4	12	24	12	12	6	24

4-tuple	5320	5311	5221	4420	4411	4330	4321	4222	3331	3322	Total
Rearrangements	24	12	12	12	6	12	24	4	4	6	286

Alternate solution: Consider the 10 pennies arranged in a linear fashion. Place three dashes anywhere between the pennies, subdividing the pennies into 4 subsets. Each arrangement of dashes forms a new grouping of the pennies. Therefore, what we need to know is how many ways three dashes can be placed separating the 10 pennies into 4 groups. Think of representing this problem as *pppppppppddd* and asking for the number of letter arrangements that can be formed. The result is

$$\frac{13!}{10!3!} = {}_{13}C_3 = 286.$$

8. $r = 1 - \sqrt[4]{\frac{40}{65}} \approx 11.43\%$.

Let r = the rate of decrease and apply the formula $A = p(1 - r)^n$ where A is the final value, p is the value after the first year, and n is the number of years of fixed reduction. If the original cost were \$100, then after the first year the value of the car is \$65. We want to know the percentage that will bring it to \$40 in four more years:

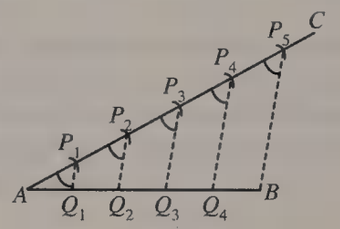
$$40 = 65(1-r)^4 \rightarrow r = 1 - \sqrt[4]{\frac{40}{65}} \approx 11.43\%$$

Alternate solution: Think of this as a geometric sequence problem. The first term is 65, and the fifth term is 40. We know that $a_n = x \cdot a_{n-1}$ and $a_1 = 65$, so $a_2 = x \cdot a_1 = 65x$; $a_3 = x \cdot a_2 = 65x^2$; $a_4 = x \cdot a_3 = 65x^3$; and

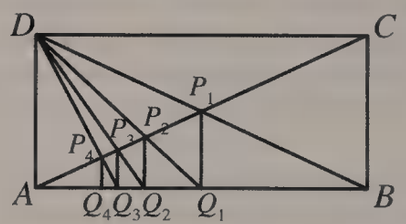
$$a_5 = x \cdot a_4 = 65x^4 = 40 \rightarrow x = \sqrt[4]{\frac{40}{65}} \approx 0.8857.$$

Thus, the percentage reduction is 1 – 0.8857 = 11.43%.

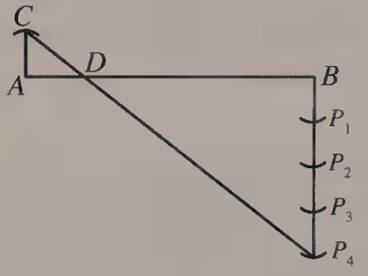
9. Euclid's procedure was to create an angle of any size using the given line segment, AB, as one part of the angle. The other ray, AC, can be of any length. Mark off five congruent segments on ray AC and name the points P_1, P_2, P_3, P_4, and P_5. Connect P_5 to point B. Construct five lines parallel to $\overline{P_5B}$ through each of the other P_n points. They intersect $\overline{AB}$ at four points, called Q_1, Q_2, Q_3, and Q_4. These four points subdivide $\overline{AB}$ into five congruent segments.



Alternate solution 1: Create a rectangle with one side as the given segment. Draw the two diagonals; from their point of intersection, construct the perpendicular to the given segment. Point Q_1 divides $\overline{AB}$ so that $AQ_1 = AB/2$. Construct $\overline{DQ_1}$. It intersects diagonal AC at P_2. Construct the perpendicular from this point to $\overline{AB}$ and label the intersection Q_2. Then $AQ_2 = AB/3$. Repeat this process two more times to get $AQ_3 = AB/4$ and $AQ_4 = AB/5$. Therefore, Q_4 is the one-fifth point. Mark off four more measures equal to AQ_4.



Alternate solution 2: Construct a line segment of any length, AC, perpendicular to the given segment at one of its endpoints. Construct a second perpendicular at the other endpoint and mark off a segment that is four times AC. Connect point C to P_4 and label the intersection with $\overline{AB}$ as point D. Thus, $AD = AB/5$.



10. 4. The problem translates to $95d - d59 = abc$ where a, b, and c are the digits 9, 5, and d in some order. Use trial and success to find the answer. Let $d - 9 = 9$; then d would have to be 8. Test the result by subtracting: $958 - 859 = 99$. So $d - 9 \neq 9$. Also, $d - 9 \neq d$. Therefore, $d - 9 = 5$, giving $d = 4$. Check to verify that $954 - 459 = 495$.

Alternate solution: Consider subtracting from the left digits first. The 9 in the $95d$ must be changed to an 8 since the remaining digits, $5d$, form a number less than or equal to the number 59, which is being subtracted from it. So $8 - d$ must be 4, 5, or 9. Obviously, it is not 9. If d were 5, then $8 - 5 = 3$, not an allowable digit in the difference. Thus, $d = 4$, and the subtraction works correctly.

11. 3. Since the area of the outer square is 12, the length of one side is $\sqrt{12} = 2\sqrt{3}$. Therefore, $AE = \sqrt{3}$. Since $\triangle AEH$ is an isosceles right triangle, $HE = \sqrt{3} \cdot \sqrt{2} = \sqrt{6}$. It follows that $EI = \sqrt{6}/2$, so $JI = (\sqrt{6}/2) \cdot \sqrt{2} = \sqrt{12}/2$. The area of the inner square is $(\sqrt{12}/2)^2 = 12/4 = 3$.

Alternate solution 1: Slide square $IJKL$ along lines HG and EF until vertex I coincides with vertex A. At this point, L and H coincide with J and E, so the area of the inner square is one-fourth the area of the outermost square, or $12/4 = 3$.

Alternate solution 2: Draw $\overline{AI}$. Prove that $AEJI$ is a parallelogram; then $IJ = AE/2$. Since the side of square $IJKL$ is one-half the side of square $ABCD$, its area is one-fourth the area of $ABCD$.

12. 4.8. Solve the system of equations by solving each for y^2 and setting the resulting expressions equal. So $y^2 = 16 - x^2$, and $y^2 = 9 - (x - 5)^2$. Thus, $16 - x^2 = 9 - (x - 5)^2$, and $x = 3.2$. Substituting this value into either of the previous equations gives us $y = \pm 2.4$ and the points of intersection P (3.2, 2.4) and Q (3.2, -2.4). So $PQ = 4.8$.

Alternate solution 1: Notice that the radii are 3 and 4 and that the distance between the centers of the circles is 5. Therefore, by connecting the two centers and one of the points of intersection, we obtain a 3-4-5 right triangle. The altitude to the hypotenuse can be found using similar triangles $h/4 = 3/5$, so $h = 12/5 = 2.4$. The distance we are seeking is twice the altitude, or 4.8.

Alternate solution 2: Use the law of cosines on the two isosceles triangles formed by pairs of radii and $\overline{PQ}$. Since two angles of the quadrilateral formed by the four radii are right angles, the other two angles are supplementary. Hence, $PQ^2 = 4^2 + 4^2 - 2(4)(4)\cos\theta = 3^2 + 3^2 - 2(3)(3)\cos(180° - \theta)$. Find that $\cos\theta = 7/25$. Substitute this result back into either equation to find that $PQ = 4.8$.

13. 2/3. Make a list of all 6 possible events; the first element in the ordered pair represents the coin selected, and the second element is the coin remaining in the bag. The possibilities are (P_1, P_2), (P_2, P_1), (N_1, N_2), (N_2, N_1), (P_3, N_3), and (N_3, P_3). If the first coin picked is a penny, there are 3 possibilities, of which 2 have the remaining coin as a penny. Thus, the probability is 2/3.

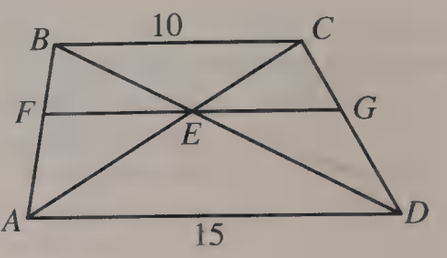
Alternate solution 1: Run an experiment in class and have students collate their results. Having each student run 10 trials should provide sufficient data to verify that the answer is 2/3.

Alternate solution 2: The formula for conditional probability is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Let $P(B)$ be the probability that the first coin is a penny and let $P(A \cap B)$ be the probability that both coins are pennies. Then $P(B) = 1/2$ since the same number of coins of each type is arranged symmetrically in the three bags, and $P(A \cap B) = 1/3$ since only one bag has 2 pennies. Finally, $(1/3)/(1/2) = 2/3$.

14. 12. Notice that $\triangle BEC \sim \triangle DEA$ with a ratio of similitude of 10/15, or 2/3. Therefore, $CE/EA = 2/3$, and $CE/CA = 2/5$. In addition, $\triangle CEG \sim \triangle CAD$, so $EG/AD = CE/CA \rightarrow EG/15 = 2/5 \rightarrow EG = 6$. Similarly, we may find that $EF = 6$, so $FG = 12$.



Alternate solution: A line parallel to the bases of a trapezoid that contains the intersection point of the diagonals has a length equal to the harmonic mean of the two bases. Therefore, FG is the harmonic mean of BC and AD, or $FG = 2(10)(15)/(10 + 15) = 12$. (The proof of this theorem follows immediately from the previous solution.)

15. 5/9. Assume that the lake is 14 miles wide. If Alf paddles for 1 hr., Bob must swim for 3 1/2 hr. to reach the kayak. At that time, the kayak has been anchored for 2 1/2 hr. Bob then paddles for 1 hr., arriving at the shore. Alf has been swimming ever since he left the kayak; that is, he has been swimming for 3 1/2 hr. and has traveled 7 miles, thus arriving at the shore at the same time that Bob reaches the kayak. In the 4 1/2 hr. that the whole process took, the boat sat idle for 2 1/2 hr. Thus, (2 1/2)/(4 1/2) = 5/9.

Alternate solution: Make a chart and examine the data (see the chart below). Thus, in 4 1/2 hr., both Alf and Bob have traveled 14 miles, and the kayak has been idle for 2 1/2 hr.

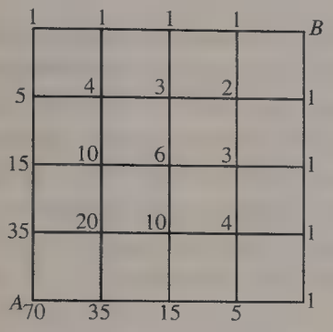
16. 3003. We must move to the right 8 times and up 6 times. Thus, what we

Time	0–1 hr.	1–2 hr.	2–3 hr.	3–3 1/2 hr.	3 1/2–4 1/2 hr.
Alf	7	2	2	1	2
Bob	2	2	2	1	7
Kayak	Used	Anchored	Anchored	Anchored	Used

need is the number of ways we can arrange 8 r's and 6 u's. This is a problem of permutations with repetition, so we can use $14!/(8! \cdot 6!) = 3003$.

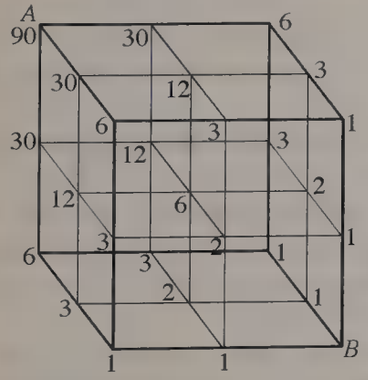
Alternate solution 1: Reaching our goal requires 14 steps, 6 of which must be upward moves. Consider this solution a combination of ${}_{14}C_6 = 3003$.

Alternate solution 2: Simplify the problem by moving A closer to B. The numbers in the array indicate the number of ways to get to B from each vertex. Enough of a pattern is displayed to relate the numbers to Pascal's triangle.



17. 90. Similar techniques used in problem 16 can be applied to this problem. Consider first getting from A to B by traveling forward twice, down twice, and to the right twice, or f, f, d, d, r, r. The number of rearrangements of these 6 letters is $6!/(2! \cdot 2! \cdot 2!) = 90$.

Alternate solution: Simplify the problem by considering A to be much closer to B and fill in all the numbers along the way. Add the numbers that have been used to count all previous positions to get the value at each vertex.



18. $(x - 3)^2 + (y - 6)^2 = 100$. The intersection of the perpendicular bisectors of any two chords of a circle is the center of the circle. Label the points A (–5, 12), B (9, 14), and C (3, –4). The midpoint $M_{AB} = (2, 13)$, and the slope $m_{AB} = 1/7$, so the equation of the perpendicular bisector of $\overline{AB}$ is $y - 13 = -7(x - 2)$. The midpoint $M_{BC} = (6, 5)$, and the slope $m_{AB} = 3$, so the equation of the perpendicular bisector of $\overline{BC}$ is $y - 5 = (-1/3) \cdot (x - 6)$. Solve these equations simultaneously to get $-7x + 27 = -x/3 + 7 \rightarrow 20x = 60 \rightarrow x = 3$. Substitute to find $y = 6$. The distance from the center (3, 6) to C (3, –4) is 10, so the equation of the circle is $(x - 3)^2 + (y - 6)^2 = 10^2 = 100$.

Alternate solution: Solve a system of three equations: $(-5 - h)^2 + (12 - k)^2 = r^2$; $(9 - h)^2 + (14 - k)^2 = r^2$; and $(3 - h)^2 + (-4 - k)^2 = r^2$. Set the first two equations equal to each other, expand, and combine like terms to get $28h + 4k = 108 \rightarrow 7h + k = 27 \rightarrow k = 27 - 7h$. Similarly, set the first equation equal to the third equation, expand, and combine terms to get $16h - 32k = -144 \rightarrow h - 2k = -9$. Substitute to get $h - 2(27 - 7h) = -9 \rightarrow h - 54 + 14h = -9 \rightarrow 15h = 45 \rightarrow h = 3$. Substitute this value back into the other equations to obtain $k = 6$ and $r = 10$.

19. 86. Let the two-digit number be $10t + u$. Solve the equation $4(10t + u) - 5(10u + t) = 4$. Simplify to get the equation $35t - 46u = 4$. Obviously, $t > u$ and, since 46 and 4 are even, t must be even. Test $t = 2$, 4, 6, and 8 and find the corresponding u value. The only possible solution is $t = 8$ and $u = 6$.

Alternate solution: Start as in the first solution but solve $35t - 46u = 4$ using mod 35. The equation becomes $24u \equiv 4 \pmod{35}$. Equivalent to 4 in mod 35 are the numbers in the set {4, 39, 74, 109, 144, ...}. Once you find a number divisible by 24, stop. So $24u \equiv 144 \pmod{35} \rightarrow u = 6$. Substitution will yield $t = 8$.

20. 1/3. Draw a vertical line through the middle of the rectangle to create three triangles, in the ratio 1:2:3, on the left of the line. The ratio of the half arrowhead to the entire left rectangle is therefore 2/6, or 1/3.

Alternate solution: Apply Pick's theorem (area = $b/2 + i - 1$, where b represents the number of lattice points on the boundary of the figure and i the number in the interior of the figure), so the area of the arrowhead is $A = 4/2 + 1 - 1 = 2$. The area of the rectangle using Pick's theorem is $A = 10/2 + 2 - 1 = 6$. Thus, the ratio is $2/6 = 1/3$.

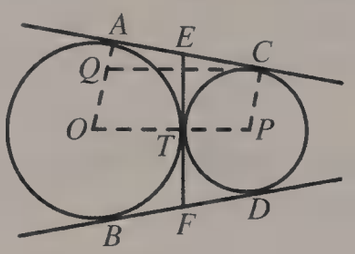
21. 0. The sine of multiples of $\pi/7$ does not yield "nice" values, so use a calculator to evaluate and sum the expressions. Make sure that the mode is set to radians. The result will probably be zero or very close to zero. The latter is a result of a rounding error.

Alternate solution: Apply the idea that the sine is an odd periodic function, so $\sin(a) = -\sin(-a) = -\sin(2\pi - a)$. Therefore, $\sin(\pi/7) = -\sin(13\pi/7)$, and $\sin(4\pi/7) = -\sin(10\pi/7)$. Since $\sin(7\pi/7) = \sin \pi = 0$, the final value for the original expression is 0.

22. 40. Label the center of the larger circle O, and label the center of the smaller circle P. Draw radii from O to A and from P to C. Draw a line segment parallel to $\overline{OP}$ passing through C and intersecting $\overline{OA}$ at Q. $OPCQ$ is a parallelogram, so $OQ = PC = 16$, $OP = QC = 25 + 16 = 41$, and $AQ = 25 - 16 = 9$. $\triangle OAC$ is a right triangle (a tangent is perpendicular to the radius at the point of tangency), so we can apply the Pythagorean theorem to find AC:

$$AC = \sqrt{41^2 - 9^2} = \sqrt{1681 - 81} = 40$$

Tangents drawn to a circle from an external point are congruent, so $AE = ET = EC = 20$. Applying symmetry principles, we find that $ET = TF = 20$. Therefore, $EF = 40$.



Alternate solution: Extend the two external tangents until they meet at a point R. By AA, $\triangle RCP \sim \triangle RTE \sim \triangle RAO$. Use the proportion $RP/16 =$

$(RP + 16 + 25)/25$ to solve for RP, which is 656/9. Next, apply the Pythagorean theorem in $\triangle RPC$ to find RC, which is 640/9. Since $EC = ET$, call them each x. Use the proportion $ET/RT = CP/RC$ to set up $x/(656/9 + 16) = 16/(640/9)$ and solve for x. The result is $x = 20$, so $ET = 40$.

23. $\left(\frac{np}{2(n+1)}\right)^2$.

Each side is s, so the short length of each rectangle will be s/n, and the perimeter will be $P = 2(s + s/n) = 2s(1 + 1/n) = 2s(n + 1)/n$. Therefore, $s = np/(2(n + 1))$, and $A = (np/(2(n + 1)))^2$.

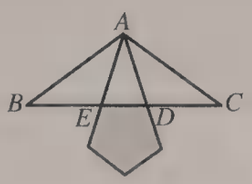
Alternate solution: The sum of all the perimeters of the rectangles = the perimeter of the square + twice the sum of all the interior segments. Thus, $np = 4s + 2(n - 1)s$, and $s = np/(2(n + 1))$.

24. 35. One way to count the cubes is to count the front faces. The top row has 1 cube, the second row has 3 cubes, and the next three rows have 6, 10, and 15 cubes, respectively (note that the number of hidden cubes in any row is equal to the number of cubes in the previous row). The sum of these numbers is 35. These numbers are known as triangular numbers; they are the sums of the counting numbers starting with 1 and are generated by the expression $n(n + 1)/2$. The sum of these numbers can be found using the expression $n(n + 1)(n + 2)/6$.

Alternate solution: Sum the numbers in each vertical stack and multiply by the number of identical stacks. We have 5 stacks of 1, 4 stacks of 2, 3 stacks of 3, 2 stacks of 4, and 1 stack of 5. The sum is $5(1) + 4(2) + 3(3) + 2(4) + 1(5) = 5 + 8 + 9 + 8 + 5 = 35$.

25. $\frac{3\sqrt{5}+7}{2} \approx 6.8541$.

Consider the top portion of the pentagon and pentagram:



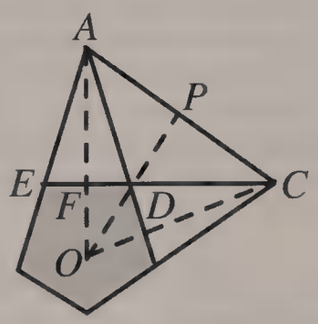
The measures of the angles of a pentagon are each 108°, so $m\angle B = m\angle C = 36°$. Since the sum of the angles of the quadrilateral formed by combining the small pentagon and $\triangle AED$ must equal 360°, $m\angle EAD = 36°$ and $m\angle CAE = m\angle AEC = 72°$. Therefore, $\overline{AD}$ bisects $\angle EAC$, and $DE/AE = DC/AC$. If we let $DE = 1$ and $AE = DC = b$, then $AC = CE = b + 1$, and $1/b = b/(b + 1)$. Thus, $1 + b = b^2$ or $b^2 - b - 1 = 0$. Apply the quadratic formula to get

$$b = \frac{1+\sqrt{5}}{2} = \varphi.$$

The ratio of the areas of the pentagons equals the ratio of the squares of their sides, so we find the ratio of the areas:

$$\frac{(1+b)^2}{1^2} = \left(1+\frac{1+\sqrt{5}}{2}\right)^2 = \left(\frac{3+\sqrt{5}}{2}\right)^2 = \left(\frac{3\sqrt{5}+7}{2}\right)$$

Alternate solution: Let O be the center of the two pentagons:



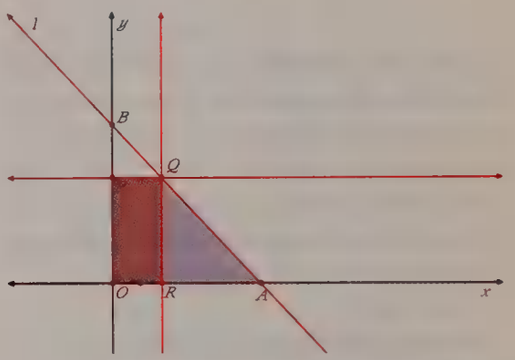
Draw $\overline{OA}$ and $\overline{OC}$ and then construct $\overline{OP} \perp \overline{AC}$. The radii of the circumscribed circle bisect the respective angles, so $m\angle CAO = 54°$ and $m\angle CAD = 36°$. Then $\triangle OPA \sim \triangle OFD$, so $OF/OD = OP/OA$. OF is the measure of the apothem of the small pentagon, and OP is the measure of the apothem of the large pentagon. If we let $AC = 2$, then $AP = 1$, $OP = \tan(54°)$, and $OA = \sec(54°)$. Using $\triangle APD$, we see that $PD = \tan(36°)$, so $OD = \tan(54°) - \tan(36°)$. Substituting all these trigonometric expressions in the proportion $OF/OD = OP/OA$ results in

$$\frac{OF}{(\tan 54° - \tan 36°)} = \frac{\tan 54°}{\sec 54°} \rightarrow$$

$$OF = (\tan 54°)(\tan 54° - \tan 36°)(\cos 54°) \approx 0.5257.$$

Similarly, $ED = (\tan 36°)(\tan 54° - \tan 36°)(\cos 36°) \approx 0.3820$. Therefore, the area of the small pentagon is $(0.5257)(10)(0.3820)/2 = 1.004$. Since $OP = \tan 54° \approx 1.3764$, the area of the large pentagon is $(1.3764)(5)(2)/2 = 6.882$. Finally, the ratio of their areas is $6.882/1.004 = 6.854$.

26. $ab/18$. The rectangle and the triangle have the same area and the same height, so the base of the triangle must be twice the base of the rectangle. Therefore, $OR = a/3$, and $RA = 2a/3$. Further, $\triangle BPQ \sim \triangle QRA$, so $BP = b/3$ and $OP = 2b/3$. Thus, the area of $\triangle BPQ = (1/2)(a/3)(b/3) = ab/18$.



Alternate solution: The ratio of the areas of similar triangles equals the ratio of the squares of a pair of corresponding sides. Thus,

$$\frac{A_{\triangle BPQ}}{A_{\triangle QRA}} = \frac{PQ^2}{RA^2} = \frac{1^2}{2^2} = \frac{1}{4}.$$

The sum of the areas of the two triangles and the rectangle equals the area of the big triangle. If we let the area of $\triangle BPQ = x$, we have $x + 4x + 4x = ab/2 \rightarrow 9x = ab/2 \rightarrow x = ab/18$.

27. 6. The sum of the angles of a triangle is always 180°, and the ratio of the angles is 5:12:13. We can solve the equation $5x + 12x + 13x = 180 \rightarrow x = 6$, so the smallest angle of the triangle is 30°. The shortest side of a triangle is

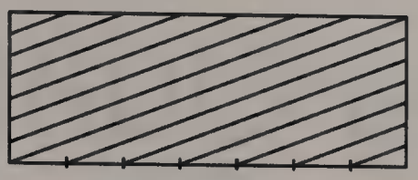
opposite the smallest angle. The 30° angle is an inscribed angle, and the central angle that intercepts the shortest side is twice the inscribed angle, or 60°. Since the sum of the central angles of all regular polygons is 360°, the polygon must have 6 central angles and, hence, 6 sides.

Alternate solution: Find the measure of the smallest angle as presented in the first solution. Since the inscribed angle is 30°, the arc it intercepts is 60°. The sum of the arcs around a circle is 360°, so the polygon must have 6 such arcs and, hence, 6 chords. Thus, $n = 6$.

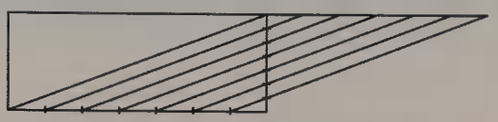
28. 128. Since $\triangle ABC$ is isosceles and $\overline{AD}$ is a median to the base, it bisects both $\angle BAC$ and $\angle BEC$. Therefore, $m\angle BED = 2 \cdot m\angle BAD$, and, since $\angle BED$ is an exterior angle to $\triangle BEA$, it equals the sum of the two remote interior angles. If we let $m\angle BED = 2x$ and $m\angle BAE = x$, then $m\angle ABE = x$ as well. Therefore, $\triangle ABE$ is isosceles, and $BE = 25$. Since $DE = 7$, by the Pythagorean theorem $BD = 24$. Use the Pythagorean theorem with $\triangle ABD$ and its legs 24 and 32 to calculate $AB = 40$. The perimeter of $\triangle ABC = 40 + 40 + 48 = 128$.

Alternate solution: Since $m\angle BEC$ is twice that of $m\angle BAC$, we can make $\angle BAC$ an inscribed angle in a circle; $\angle BEC$ will be the central angle intercepting the same arc, $\overset{\frown}{BC}$. This implies that $\overline{BE}$ is a radius, and so its length must be 25, the same as AE. If we extend $\overline{AD}$ so that it intersects the circle at point P, we see that $DP = 18$. We can calculate BD using the theorem that when two chords intersect inside a circle, the product of the segments of one chord equals the product of the segments of the other chord. Thus, $32 \cdot 18 = x^2 \rightarrow x = 24$. We can calculate AB in the same way as in the first solution.

29. 91. Since the rectangle has sides 5 and 12 and each side is divided into 7 congruent segments, the length of each segment is either 5/7 or 12/7. The smallest triangle in the upper-left corner of the diagram has a hypotenuse of 13/7. Each successive hypotenuse is a multiple of 13/7—26/7, 39/7, ..., 78/7—and the main diagonal is 91/7 = 13. Each length is duplicated in the second portion of the figure, so the total of all the lengths will be $2(13/7)(1 + 2 + 3 + 4 + 5 + 6) + 13 = 91$.



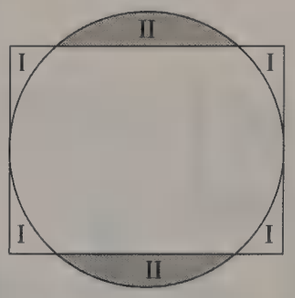
Alternate solution: Translate the upper-left triangular region of the diagram to form the parallelogram in the diagram. The diagonal of the rectangle is 13, and there are 7 such lengths, so the total length is $7 \cdot 13 = 91$.

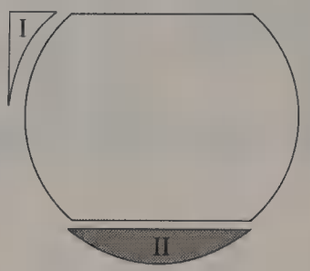


30. 1106. Apply the formulas $a_n = a_1 + (n - 1)d$ and $S_n = n(a_1 + a_n)/2$ to find the 1st term, the 80th term, and the sum of the terms $a_2 + a_4 + a_6 + \cdots + a_{80}$. The sequence is arithmetic, so $a_2 = a_1 + 5$; $a_3 = a_1 + (2)5$; and $a_{80} = a_1 + (79)5$. Since the sum of the 80 terms is 2012, we have $(80/2)(a_1 + a_1 + 5(79)) = 2012$, or $a_1 = -172.35$. Next, determine a_{80}, which is the 40th term of the requested sum. Let $a_2 = b_1$ and $a_{80} = b_{40}$; then $b_{40} = -167.35 + 39(10) = 222.65$. So $S_{40} = (40/2) \cdot (-167.35 + 222.65) = 1106$.

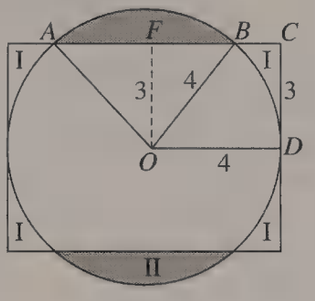
Alternate solution: Rewrite the series as $(a_1 + a_3 + a_5 + \cdots + a_{79}) + (a_2 + a_4 + a_6 + \cdots + a_{80}) = 2012$ and then substitute for the following: $a_1 = a_2 - 5$; $a_3 = a_4 - 5$; $a_5 = a_6 - 5$; ... ; and $a_{79} = a_{80} - 5$. Thus, $(a_2 + a_4 + a_6 + \cdots + a_{80} - 40(5)) + (a_2 + a_4 + a_6 + \cdots + a_{80}) = 2012 \rightarrow 2(a_2 + a_4 + a_6 + \cdots + a_{80}) - 200 = 2012$, so $a_2 + a_4 + a_6 + \cdots + a_{80} = (2012 + 200)/2 = 1106$.

31. 2.27. The difference between the two areas is the difference between the area of the circle and the area of the rectangle, or $\pi(4)^2 - (6)(8) = 16\pi - 48 \approx 2.27$. The circle consists of a region common to both the circle and the rectangle plus two circle segments (II), and the rectangle consists of the common region and four right-triangular regions with a curved hypotenuse (I). The difference between the portion inside the circle but outside the rectangle and the portion inside the rectangle but outside the circle is simply the difference between the two circle segments and the four right-triangular regions.



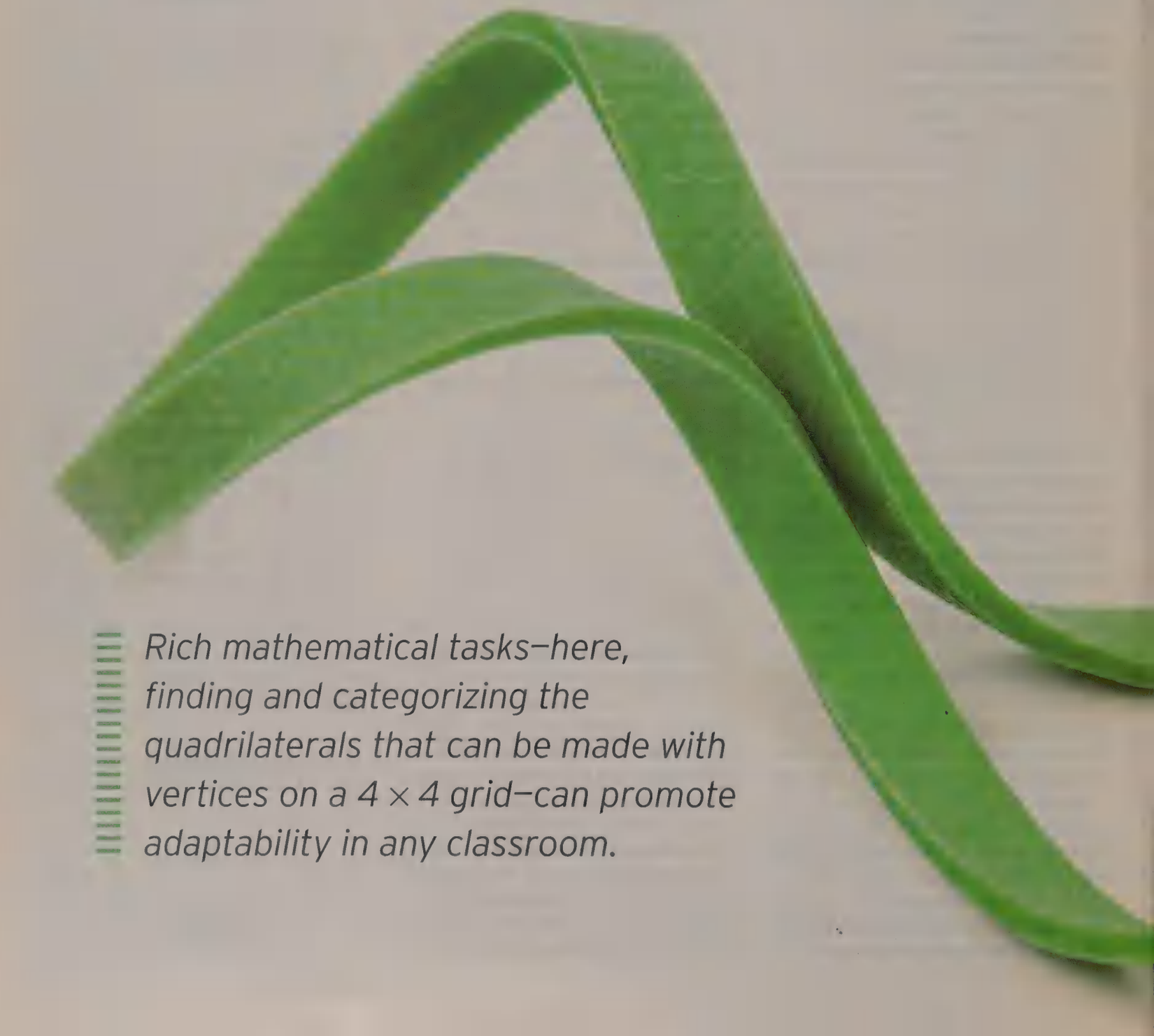


Alternate solution: Calculate the area of the circle segments (parts outside the rectangle but inside the circle) and then calculate the area of the right-triangular regions (parts inside the rectangle but outside the circle) and then subtract. The area of the circle segments is found by subtracting the area of $\triangle AOB$ from the area of sector AOB. We know that $OF = 3$ and $OB = 4$, so $FB = \sqrt{7}$, and the area of $\triangle AOB = 3\sqrt{7} \approx 7.937$. Then $m\angle AOB = 2\arccos(3/4) \approx 82.819°$, so the area of sector $AOB \approx \pi(82.819/360) \cdot (16) \approx 11.564$. Therefore, the area of circle segment $AB \approx 3.627$. The area of one right-triangular region is the area of trapezoid $ODCB$ minus the area of sector BOD. We know that $BC = 4 - \sqrt{7}$, $OD = 4$, and height $CD = 3$, so $A_{trap} = (3/2)(4 + 4 - \sqrt{7}) = 12 - 1.5\sqrt{7} \approx 8.031$. Then $m\angle BOD = 90° - \cos^{-1}(3/4) \approx 48.590°$, so the area of the sector is $\pi(48.590/360)(16) \approx 6.784$. Subtracting the two areas results in $8.031 - 6.784 = 1.247$. Finally, the difference between the areas of the two circle segments and the four right-triangular regions is $2(3.627) - 4(1.247) = 2.266 \approx 2.27$.



Exploring Quadrilaterals

Flexible A



Rich mathematical tasks—here, finding and categorizing the quadrilaterals that can be made with vertices on a 4 × 4 grid—can promote adaptability in any classroom.

through pproaches

Kerri Richardson, Anne Reynolds, and Catherine S. Schwartz

The quadrilaterals problem is a particularly rich task, applicable to in-service teachers as well as to the students they teach:

> Given the set of sixteen dots shown (see **fig. 1**), draw all possible four-sided figures with corners at the dots. Develop a system for knowing when you have identified them all. Classify the quadrilaterals in as many ways as you can. Example: Classify by the number of right angles. What are the possible perimeters? What are the possible angle measures?

The ways in which students engage in this task, the strategies they use, and the mathematics that results align well with this *MT* Focus Issue on flexible mathematical thinking. We offer our observations of various groups of in-service teachers as they investigate this problem and then extend these insights by showing how middle-grades students approached the same task. The purpose of sharing these experiences is to expose the rich potential of the task and to highlight the flexible nature of students' thinking about the task.

The ideas presented here grew out of a presentation at an NCTM annual meeting and an article written from a theoretical perspective (Richardson, Reynolds, and Stein 2008; Richardson, Schwartz, and Reynolds 2010). We have also benefited from the works of Kennedy (1993), who looked at triangles on an array, and Kennedy and McDowell (1998), who provided the actual number of quadrilaterals on a 4×4 array.

DEVONYU/ISTOCKPHOTO

THE STUDENTS

Our graduate-level mathematics education students are predominantly a combination of in-service elementary, middle, and high school teachers. They often take the quadrilaterals task to their own classrooms and use it productively with their students, demonstrating that the task is suitable for students at various levels. In this on-going investigation, in-service teachers work alone or in groups of two, both in and outside class. At the end of the semester, pairs or small groups of teachers present their solutions to the entire class, usually in the form of PowerPoint® presentations but sometimes in other forms, such as large sheets of paper with various illustrations. The teachers are asked to describe and discuss the strategies that they used and developed over the course of the semester. What follows is a look at some of the solutions presented by in-service teachers and some discussion of how their own students investigated the task.

THE QUADRILATERALS PROBLEM

Task and Goal Clarification

What counts as an acceptable solution? One of the first questions posed by in-service teachers is, "Does a different location on the array count as a different solution?" (see **fig. 2**). The consensus is that it does not. Students must focus on two major goals of the problem: (1) to identify and count all *distinct* quadrilaterals that can be formed on a 4×4 grid and (2) to classify each quadrilateral they discover.

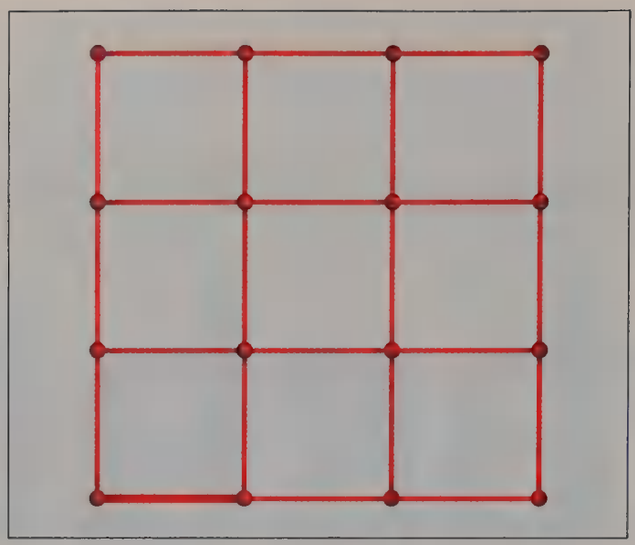


Fig. 1 How many quadrilaterals can be found?

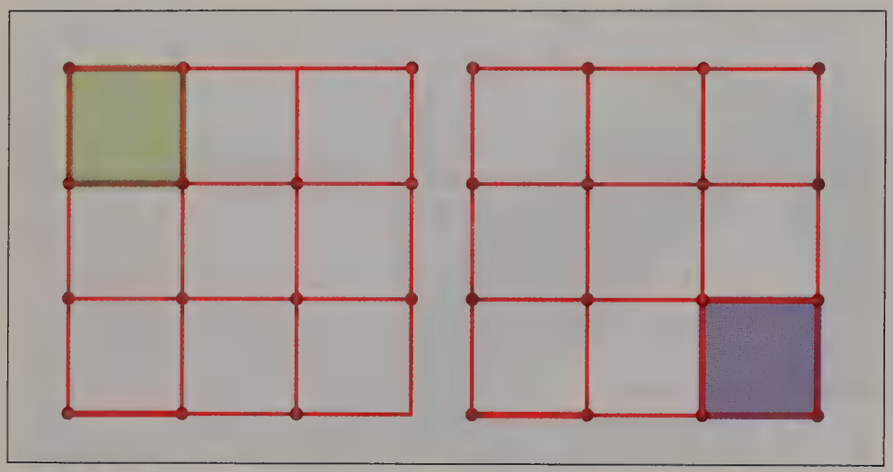


Fig. 2 A decision had to be made—one or two quadrilaterals?

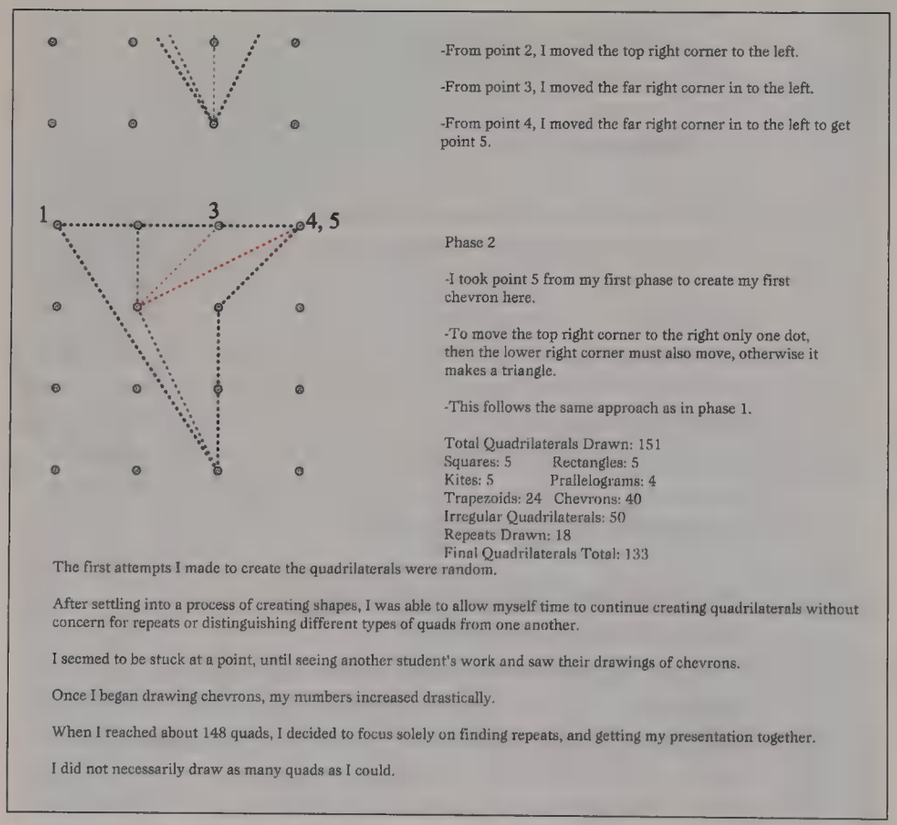

-From point 2, I moved the top right corner to the left.
-From point 3, I moved the far right corner in to the left.
-From point 4, I moved the far right corner in to the left to get point 5.
1
3
4, 5
Phase 2
-I took point 5 from my first phase to create my first chevron here.
-To move the top right corner to the right only one dot, then the lower right corner must also move, otherwise it makes a triangle.
-This follows the same approach as in phase 1.
Total Quadrilaterals Drawn: 151
Squares: 5 Rectangles: 5
Kites: 5 Prallelograms: 4
Trapezoids: 24 Chevrons: 40
Irregular Quadrilaterals: 50
Repeats Drawn: 18
Final Quadrilaterals Total: 133
The first attempts I made to create the quadrilaterals were random.
After settling into a process of creating shapes, I was able to allow myself time to continue creating quadrilaterals without concern for repeats or distinguishing different types of quads from one another.
I seemed to be stuck at a point, until seeing another student's work and saw their drawings of chevrons.
Once I began drawing chevrons, my numbers increased drastically.
When I reached about 148 quads, I decided to focus solely on finding repeats, and getting my presentation together.
I did not necessarily draw as many quads as I could.

Fig. 3 This work sample provides a glimpse into this teacher's thinking as she progressed through the investigation.

Organization of Materials

A second common concern is how to organize the quadrilaterals. Important materials to have on hand include dot paper, geoboards, and computers, if available. Online geoboards that allow groups to place virtual rubber bands have proven to be quite helpful as a starting point or even as a way to check various quadrilateral duplicates efficiently. For many in-service teachers, the focus on developing a systematic way to keep track of the quadrilaterals is vital. The idea that a formula could be found for the number of all possible quadrilaterals emerges quite soon for the majority of students. Interestingly enough, however, these early starting points are almost always abandoned in pursuit of a more comprehensive or efficient approach.

In the following sections, we present work samples from our in-service teachers. Previously we have detailed similar ideas but from philosophical perspectives (see Richardson, Schwartz, and Reynolds 2010).

Orientation Strategies

Orientation of the quadrilaterals by shape on the array is the primary approach used by our students. One high school teacher developed a system of orienting the quadrilateral using three fixed points (see **fig. 3**) and then considered all the different types of quadrilaterals constructed using these points. As a result of engaging with others in the class about differing approaches, she realized that chevrons also needed to be a part of the quadrilaterals.

Another orientation strategy involved simultaneously fixing the number of dots or pegs and then systematically constructing all possible quadrilaterals with the selected dots or pegs. The example shown in **figure 4**, which was completed by two high school teachers, demonstrates that as the number of dots in the interior and on the perimeter of the quadrilateral increases, the variety and number of quadrilaterals also increase. A constraint of this approach relates to which fixed dots are selected in each case; selection appears to make a difference when counting distinct quadrilaterals.

A refinement of the approach shown in **figure 3** involved organizing the search by rows. **Figure 5**, which was completed by a middle-grades teacher, illustrates the beginning of this process using dots in the first two rows as anchors for the quadrilaterals. The process would be completed by using the first three rows and then all rows.

Some in-service teachers stay with their orientation strategy and see it through until they have exhausted what they believe to be all the quadrilaterals. Part of such a process involves noticing the duplicates that emerge. When teachers observe duplicates, the problem within the problem frequently emerges. Their attention then shifts to

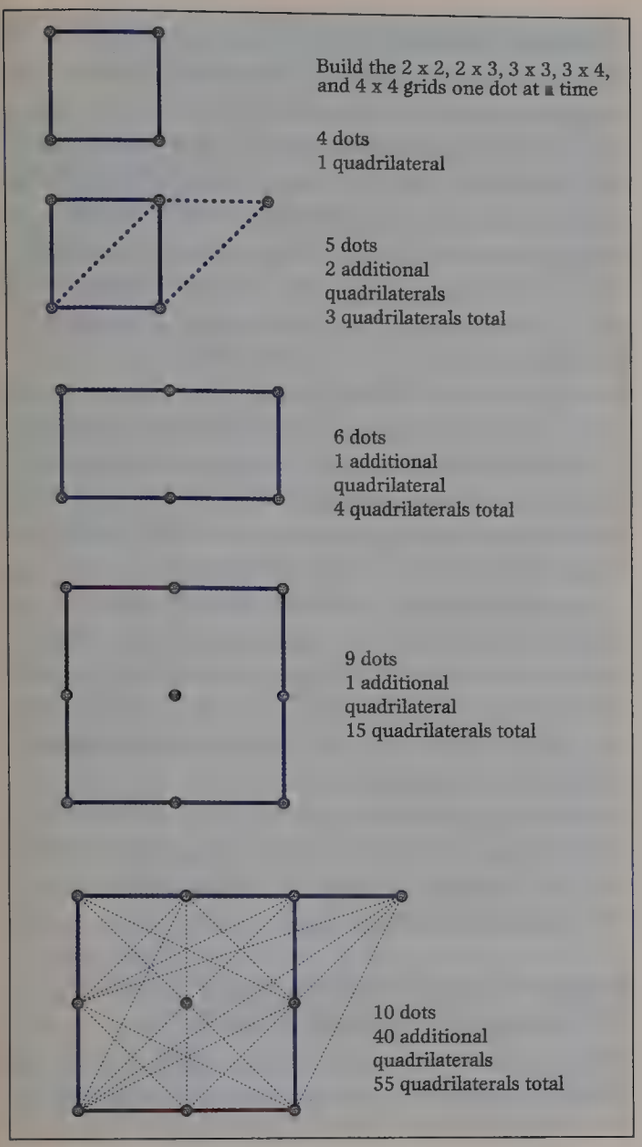

Build the 2 x 2, 2 x 3, 3 x 3, 3 x 4, and 4 x 4 grids one dot at a time
4 dots
1 quadrilateral
5 dots
2 additional quadrilaterals
3 quadrilaterals total
6 dots
1 additional quadrilateral
4 quadrilaterals total
9 dots
1 additional quadrilateral
15 quadrilaterals total
10 dots
40 additional quadrilaterals
55 quadrilaterals total

Fig. 4 Some students built up the complexity one step at a time.

eliminating duplicates rather than generating new quadrilaterals. The process for deleting duplicates has involved strategies such as laying clear overhead projection paper over dot paper, tracing the quadrilateral, and then rotating the paper to see whether the traced quadrilateral is the same as another one.

The tedium of this strategy drives some teachers to move on to other approaches, such as focusing on the area and perimeter of the shapes that emerge. Thus, the challenge of identifying and tracking duplicates leads to a multipronged approach that uses various orientation strategies, along with area and perimeter of the quadrilaterals. This is the point at which the immense flexibility between approaches becomes clear.

Measurement Strategies

Two other high school teachers who worked together used a perimeter approach. They began by identifying and documenting the types of quadrilaterals they found. Their knowledge of the Pythagorean theorem, however, moved their focus to the

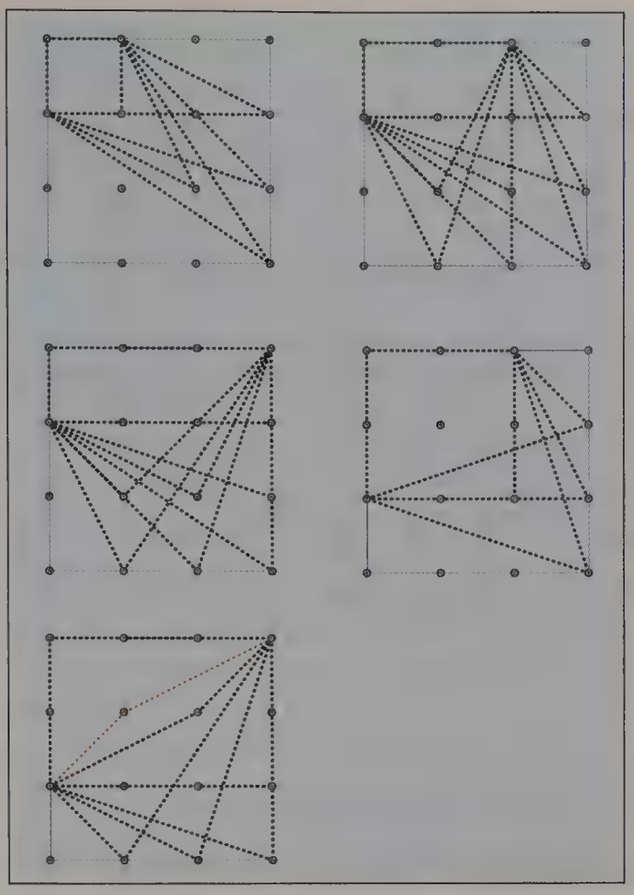


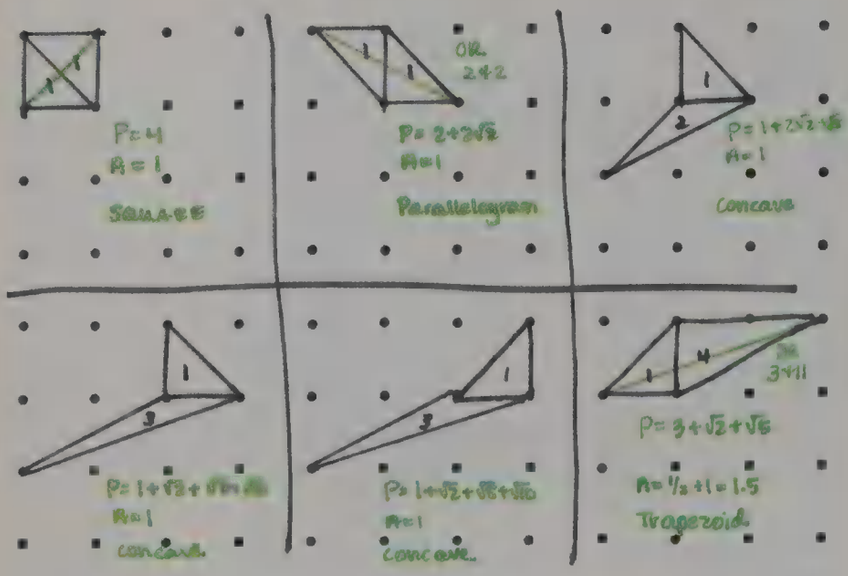
Fig. 5 One teacher used points and rows to anchor sets of quadrilaterals.

length of the sides of the quadrilaterals, and they came up with numbers involving square roots for possible side lengths and perimeters. The exploration became a more specific investigation of the perimeter of the quadrilaterals.

Another in-service high school teacher demonstrated a combination of orientation, perimeter, and area strategies (see **fig. 6a**). This teacher allowed the orientation strategy to drive her work, but she soon started analyzing the sides and areas of shapes. She began with triangles because she found them to be more manageable (again, we see evidence of the flexibility of the task) and went on to use the triangles to form quadrilaterals, as shown in the work sample. Her presentation was on a large poster board; **figure 6b** shows some processes that she used. Again, we can see how the very nature of the investigation took a variety of twists and turns for this teacher and how the focus, rather than the final answer, became the approach or the strategy. Another possible strategy (one that we did not observe in use) might be combining quadrilateral classification with both area and perimeter computations to demonstrate uniqueness.

Similarities and Differences between Approaches

In-service teachers almost always identify quadrilaterals by shape. The next step is to attempt a classification that leads from small to larger



(a)

Where Have I Been?

- First approach was very straightforward: find all quads systematically.
- Then I noticed that the quads were in fact just triangles.
- Before finding all possible triangles, I spent several weeks trying to find an equation whose answer would answer the question, "How many quads are there?"
- Painstakingly I have set out to find all combinations of triangles that form a quadrilateral.

Where Can I Go?

- My first instinct would be to first recreate the entire process again. If I arrive at the same answer twice, then I would be more confident in my answer.
- Next, I could reorganize all the quads by their area and the triangles that are being used.
- Or I could reorganize all the quads by their most specific title. Then I could look for patterns with the type of quad and the triangles that are being used.

(b)

Fig. 6 One teacher used triangles to organize her approach (**a**) and her presentation (**b**).

quadrilaterals. Materials such as geoboards and transparency paper help. Generally, however, differences begin to emerge as a result of the participants' mathematical backgrounds.

For many K–8 teachers, orientation strategies become the primary focus of the task. For 9–12 teachers, the measurement strategies require knowledge of the Pythagorean theorem as well as the ability to work with irrational numbers; thus, they may use measurement strategies later, when they exhaust the usefulness of the orientation strategies. Both measurement and orientation strategies may also be used in conjunction with one another.

Unexpected Approaches

On two occasions, we have been surprised by unique approaches to the task.

In one class, a cooperating pair of teachers who had generated a large number of quadrilaterals were having difficulty sorting through them to identify and eliminate duplicates. None of the methods that they tried proved satisfactory. The teachers, knowing that vectors had something to do with directed line segments, surmised that they might use them as a way to identify duplicates. Although they were never able to make vectors work for them in their analysis of the various quadrilaterals, during subsequent weeks they learned a considerable amount about something that was for them a whole new field of mathematics—vector analysis.

In another class, two doctoral students took another noteworthy approach inspired by a statistics course they were taking together at the time. They connected their experiences in the class with the degrees-of-freedom concept (see **fig. 7**). If two of the sides of a quadrilateral were set, they reasoned, once they drew the third side, the fourth side would be determined. Therefore, their strategy was to draw all the possible quadrilaterals when two adjacent sides were fixed.

SHARING THE QUADRILATERALS TASK WITH MIDDLE-GRADES STUDENTS

When our in-service teachers have used this task with their students, they have found that these students can explore the task at different levels, depending on their mathematical development.

For students in grades 4–8, the quadrilaterals task connects well with much of the mathematics that they are just making sense of, particularly in their geometry, measurement, and number work. With younger students, the activity has encouraged the exploration of the different types of quadrilaterals. The students quickly find the regular quadrilaterals (squares) and then all possible rectangles and then branch out from there to other classes of quadrilaterals. An interesting result is that the shapes generated are usually all convex quadrilaterals. Here the geoboards that some students use become useful. As students stretch their rubber bands around pegs, some get caught unintentionally on other pegs, resulting in a quadrilateral that is not convex; should it count? After some discussion, the class agrees that indeed it should count, and the hunt begins for these "odd" shapes.

One pair of students decided to classify their quadrilaterals by the number of right angles formed. They began by getting five fresh sheets of dot paper and giving them headings such as these: "0 right ∠s," "1 right ∠," "2 right ∠s," "3 right ∠s," "4 right ∠s" (meaning exactly 0, 1, 2, 3, or 4 right angles).

They then began to organize their shapes, starting with the shapes they had found with no right angles. When the students reached the sheet headed "3 right ∠s," they were puzzled because they could not find any shapes to meet this criterion. They experimented for some time and decided that every quadrilateral that had three right angles had to have four. This conjecture led to an interesting class discussion about angle measure in quadrilaterals.

In trying to find the area of some of their quadrilaterals, students have to identify a unit, because none is specified. Usually, they define a *unit* as the distance between horizontally or vertically adjacent points on the grid. This definition leads to an interesting dilemma because students often label the distance between two adjacent points along a diagonal as 1 unit. Others go straight to labeling a square formed by joining four adjacent points as a "square unit." When they connect diagonally adjacent points to make a square, they frequently claim that it is also a square unit (see **fig. 8**). At this point, some groups of students are cutting out the shapes they had drawn on the square grid paper to categorize them and also to check for duplicates (by placing shapes on top of one another). Then they notice that these shapes do not have the same perimeter or area.

With one group of sixth-grade students, the in-service teacher encouraged them to find the length of the side of this square made on the diagonal. Students used the side of the unit square to measure off this length and found that it was more than 1 unit but less than 1 1/2 units; further investigation led them to find that it was between 1 1/4 and 1 1/2 units and then between 1 1/3 and 1 1/2 units.

At this point, students asked how they could get a more "exact" measurement. The teacher took the opportunity to introduce the Pythagorean theorem to the class, and students used class computers to explore visual animations of various proofs of this theorem. They then began forming right triangles to calculate various diagonal lengths on the 16-point grid. They found that the length of this "diagonal square" (as they called it) was $\sqrt{2} \approx 1.4142$. To check this result against their earlier prediction that the length was between 1 1/3 and 1 1/2, they converted these fractions into decimals—1.33 and 1.5—and convinced themselves that their original prediction was good. For a diagonal length that extended over three dots, some students measured this as $2\sqrt{2}$ (reasoning that connecting two dots diagonally was $\sqrt{2}$ and that a segment connecting three dots was just double that length), whereas other students used the Pythagorean theorem to find this length to be $\sqrt{8}$, or 2.8284. The teacher challenged students to convince themselves that $2\sqrt{2}$ and $\sqrt{8}$ were the same without using decimals. This challenge led to an exploration of how to simplify square roots.

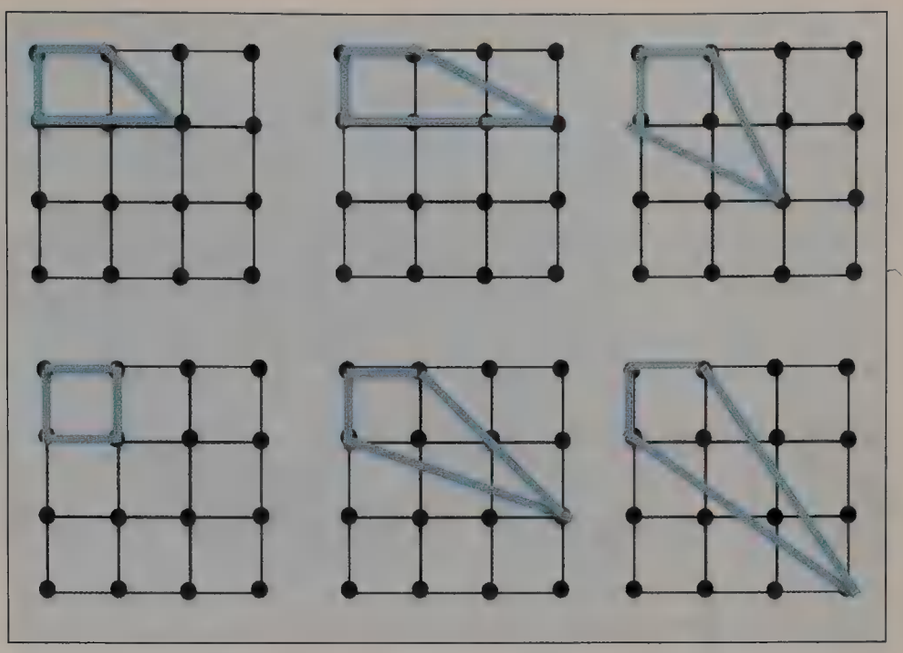


Fig. 7 Students taking a statistics course contemporaneously found a unique approach.

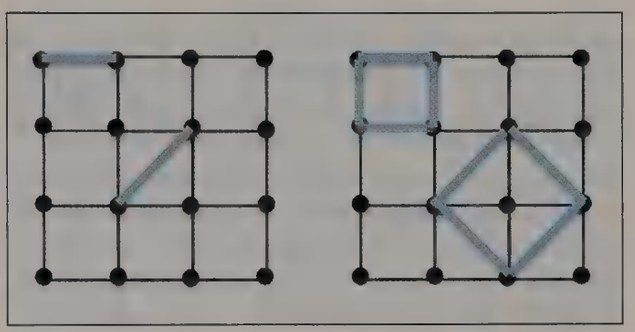


Fig. 8 Some students thought that both segments were units and that both squares were unit squares.

DISCUSSION

The quadrilaterals task has proven to be rich in mathematical ideas. In-service teachers used both geometric and measurement concepts in various problem-solving strategies in their attempts to solve this problem. The geometric approach often results in an exploration of spatial concepts, angles, transformations, congruence and similarity, and concave and convex shapes. The measurement approach frequently takes into account variable concepts, area and perimeter, the Pythagorean theorem, linear and square units, and rational and irrational numbers (to name a few).

With respect to using all strategies or viewing the similarities of other solutions, teams or individual students generally do not use all strategies but almost always share their work as the investigation progresses. This sharing allows the in-service teachers to appreciate one another's thinking and gain additional insight into the mathematics. The solutions in essence provide a means for organizing the same information (for example, students may look at two quadrilaterals and intuitively guess that one has greater perimeter than the other, or perhaps the

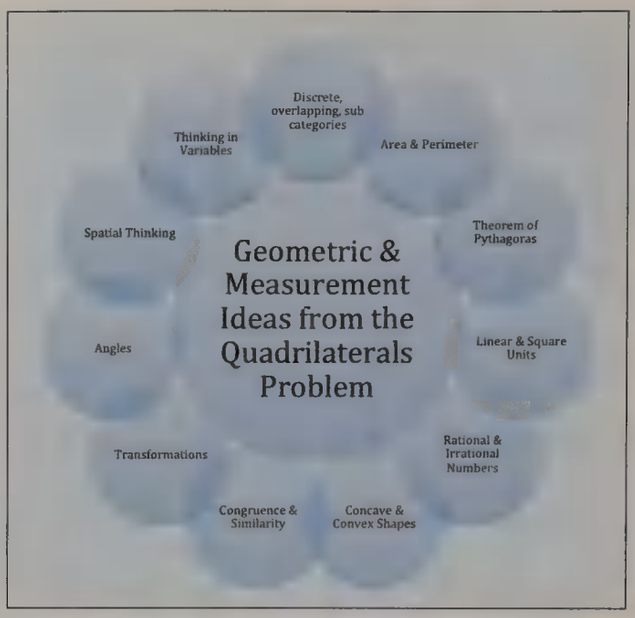


Fig. 9 The quadrilaterals task encompasses a great deal of mathematics, both on the surface and below it.

teacher can also provide string and students can use that to measure perimeter, thus avoiding the irrational number computations). When in-service teachers were faced with a challenge (i.e., looking at two quadrilaterals that looked as if they had the same perimeter), they tried to come up with a resolution. They might consider this question: Is there a way to get an accurate measure of the size? This question then leads to the Pythagorean theorem.

Figure 9 may be a useful tool for a teacher who is planning the activity in class because it conveys the fluidity of both approaches, which is important in representing all student experiences with the problem. All students need this type of flexibility when investigating such a rich task because they often move between and combine multiple approaches. In addition, there is an opportunity to discuss with students that this is what mathematicians do all the time—explore new ideas, make conjectures, argue about their latest findings, and formalize their work.

As the mathematician John H. Conway stated on the *Nova* episode devoted to Andrew Wiles's proof of Fermat's last theorem (Lynch and Singh 1997): "I'm sad in some ways, because Fermat's last theorem has been responsible for so much. What will we find to take its place?" Connected tasks (Richardson, Carter, and Berenson 2010) such as this quadrilaterals investigation give prospective and practicing teachers a chance to experience for themselves the value of learning mathematics through problem solving (Hiebert 1999; Wheatley 1991).

ACKNOWLEDGMENTS

The authors wish to thank Preety Tripathi of the Mathematics Department at SUNY at Oswego for her help in editing this article as well as all the students who worked so diligently on this problem.

REFERENCES

Hiebert, James. 1999. "Relationships between Research and the NCTM Standards." *Journal for Research in Mathematics Education* 30 (1): 3–19.

Kennedy, Joe. 1993. "Problem Solving on Geoboards." *Mathematics Teacher* 86 (1): 82.

Kennedy, Joe, and Eric McDowell. 1998. "Geoboard Quadrilaterals." *Mathematics Teacher* 91 (4): 288–90.

Lynch, John (writer and producer), and Simon Singh (director). 1997. "The Proof," an episode of *Nova*. Paula Apsell, senior executive producer. Boston: BBC TV/WGBH Boston Co-Production.

Richardson, Kerri, Tyrette Carter, and Sarah Berenson. 2010. "Connected Tasks: The Building Blocks of Reasoning and Proof." *Australian Primary Mathematics Classroom* 15 (4): 17–23.

Richardson, Kerri, Anne Reynolds, and Catherine Stein. 2008. "Investigating Quadrilaterals on a Four-by-Four Array." Gallery workshop conducted at the National Council of Teachers of Mathematics Annual Meeting in Salt Lake City, Utah.

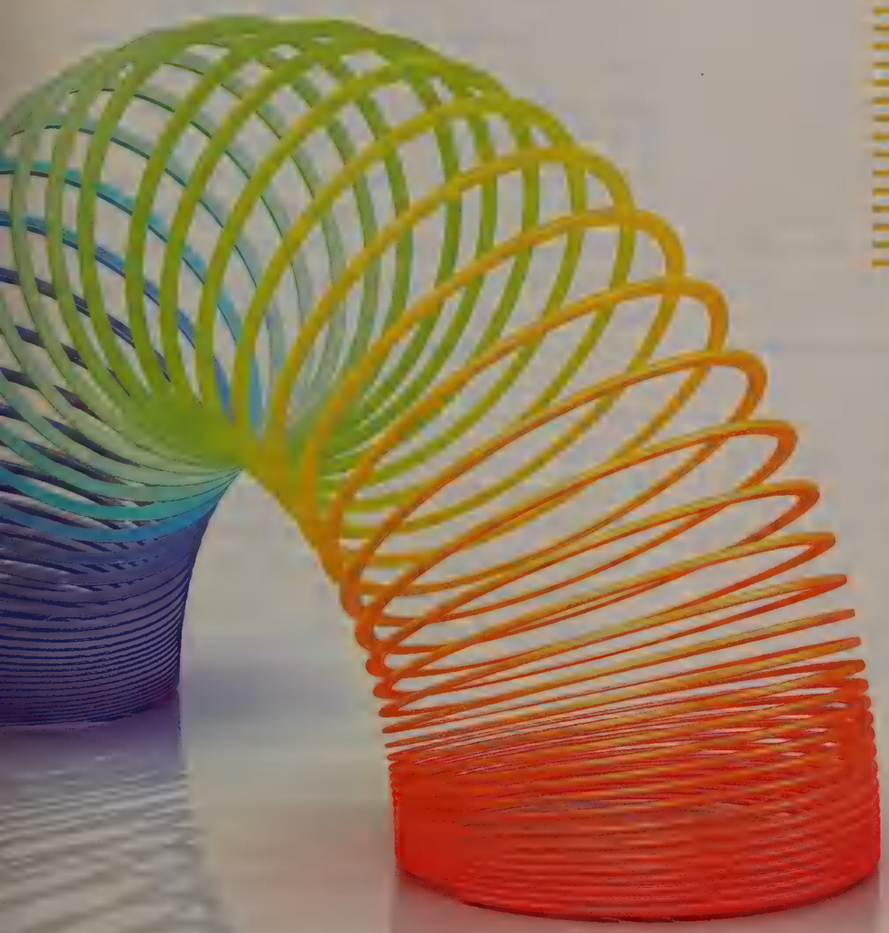
Richardson, Kerri, Catherine S. Schwartz, and Anne Reynolds. 2010. "Investigating Quadrilaterals as an Ongoing Task." *International Journal of Mathematics Teaching and Learning* (October 12): 1–21. http://www.cimt.plymouth.ac.uk/journal/default.htm.

Wheatley, Grayson H. 1991. "Constructivist Perspectives on Mathematics and Science Learning." *Science Education* 75 (1): 9–21.

KERRI RICHARDSON, kerri_richardson@uncg.edu, is an associate professor of mathematics education at the University of North Carolina at Greensboro. She is interested in how teachers use representations to improve students' understanding of mathematical content. ANNE REYNOLDS, areynolds@ncsu.edu, is an associate professor of mathematics education at North State University in Elon. She is interested in how students learn mathematics and in supporting their learning through problem-solving activities. CATHERINE S. SCHWARTZ, schwartzcs@ecu.edu, is an assistant professor of mathematics education at East Carolina University in Greenville, North Carolina. She is interested in the ways in which elementary teachers develop their mathematics teaching over time.

Looking for Connections between Linear and Exponential Functions

Jane-Jane Lo and
James L. Kratky



Students compare and contrast linear and exponential relationships from tabular, graphical, and algebraic representations.

The linear relationship is the first functional relationship that many students encounter both inside and outside mathematics classes. As soon as young children are introduced to the magic of counting, they are eager to recite the sequences 1, 2, 3, 4, 5, …; 2, 4, 6, 8, 10, …; 5, 10, 15, 20, 25, …; and so on. However, the formalization of this relationship through symbols and notation as well as the correct use of this relationship in solving problems present significant challenges for students (Lee and Freiman 2006). Many are given the formula $y = mx + b$ too quickly to allow proper development of the concept of rate of change—that is, slope (Teuscher and Reys 2010).

SASHKINW/ISTOCKPHOTO

Similar results occur when students investigate exponential relationships. Often, the knowledge gaps from linear functions bleed into students' understanding of exponential relationships. Students frequently have difficulty determining whether a given real-life situation is best modeled as a linear relationship or as an exponential relationship. One root of such difficulty is the lack of deep understanding of the very concept of *rate of change*: The rate of change in a linear relationship is constant, whereas the rate of change in an exponential relationship $y = a(b)^x$ increases when $b > 1$, decreases when $0 < b < 1$, or remains constant when $b = 1$.

We will provide a lesson that allows students to reveal their misconceptions and improve their understanding of this significant concept as they explore the structural connections between these two mathematical relationships. Students will develop flexible thinking by comparing the structure of linear and exponential relationships and translating fluently between and within representations. "Look for and make use of structure" is one of the eight Standards for Mathematical Practice put forth in the Common Core State Standards for Mathematics (CCSSI 2010). We encourage students to look across and within various representations and establish meaningful connections that aid them in successful problem solving across different mathematical terrains (Cuoco, Goldenberg, and Mark 2010).

The lesson focuses on tabular, graphical, and algebraic representations in a way that allows time for rich discussions and debate. The lesson is best used after students have some experience working with linear and exponential relationships separately, typically toward the end of a first-year algebra class or an integrated mathematics class. The lesson overview (see **fig. 1**) allows students to follow various paths during implementation while still attending to each part of the lesson.

The lesson ideas are situated around a single task, the Spanish word task (see **fig. 2**), adapted from course 1 of the Core Plus Mathematics Project (Coxford et al. 2003, pp. 119–20).

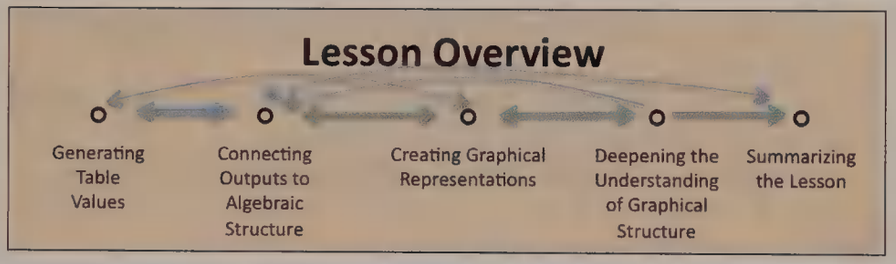


Fig. 1 This multiday lesson design also allows for iterative cycles in which students may engage as they practice mathematical modeling.

Suppose that when school ends in June, you know the meaning of 500 Spanish words but that you will not study or speak Spanish during the summer vacation. Consider the following two models for memory loss.
Model 1: You will forget 20 words each week.
Model 2: During each week of summer, you will forget 5% of the words you know at the beginning of that week.
(*a*) Compare and contrast the patterns in tabular, graphical, and algebraic representations of these two models.
(*b*) Which of the two is the better model and why?

Source: Coxford et al. 2003, pp. 119–20

Fig. 2 The Spanish word task provides rich opportunities for comparing linear and exponential growth or decay.

GENERATING TABLE VALUES

Students who are not sure where to start should be encouraged to generate a table to compare the outputs between these two models. The number of words left in the memory through week 7 of the summer is shown (to the nearest whole number) in **table 1**.

Allow students to collaborate in groups. Observe them as they work and record the ideas that emerge from their collaborations. Look for groups that identify the additive (subtraction) relationship in model 1 and the multiplicative relationship in model 2. These patterns transfer naturally to recursive formulas that can be used to graph the functions later: $Next = Now - 20$ for model 1 and $Next = Now \cdot (0.95)$ for model 2. Some students may jump right to the explicit equations, but this is not yet the focus. Make a note of students who provide unique yet still correct approaches to the tabular part of the task. Observed differences between groups and individuals can be a great conversation booster when managing a whole-class conversation.

A good way to open a whole-class discussion is to start with tabular representation. A primary reason for doing so is that many students will be able to generate the tables efficiently (or get quick help from a friend). This is a point at which students' misconceptions can be glossed over or covered up. Viable questions for eliciting student thinking include these:

- What are some similarities and differences between the two sets of table values?
- Why does model 2 show a decrease of 25 words between weeks 0 and 1 but a decrease of about 24 words between weeks 1 and 2?
- Will the values of model 2 always be less than the values of model 1? Why or why not?

These questions begin to expose what is happening in each model and set the stage for a deeper consideration of mathematical structure. Allow students time to think and respond to one another. If the conversation stalls, point out differences that you have observed between different groups and challenge students to rectify, explain, or dispute the

Table 1 Loss of Vocabulary

Week in summer	0	1	2	3	4	5	6	7
Words in memory (model 1)	500	480	460	440	420	400	380	360
Words in memory (model 2)	500	475	451	429	407	387	368	350

Table 2 Arithmetic Patterns

Input	Model 1	Model 2
0	500	500
1	$500 - 20$	$500 - 0.05(500) = 500(0.95)$
2	$500 - 20 - 20 = 500 - 2(20)$	$500(0.95)(0.95) = 500(0.95)^2$
3	$500 - 2(20) - 20 = 500 - 3(20)$	$500(0.95)2(0.95) = 500(0.95)^3$
4	$500 - 3(20) - 20 = 500 - 4(20)$	$500(0.95)3(0.95) = 500(0.95)^4$
5	$500 - 4(20) - 20 = 500 - 5(20)$	$500(0.95)4(0.95) = 500(0.95)^5$
6	$500 - 5(20) - 20 = 500 - 6(20)$	$500(0.95)5(0.95) = 500(0.95)^6$
7	$500 - 6(20) - 20 = 500 - 7(20)$	$500(0.95)6(0.95) = 500(0.95)^7$
...	...	...
n	$500 - (n - 1)(20) - 20 = 500 - n(20)$	$500(0.95)^{n-1}(0.95) = 500(0.95)^n$

differences. Encourage students to pose their own questions to one another. If the conversation seems to burn out too quickly, probe further. Ask students to think about what would happen if the patterns continued on beyond week 7 (that is, if the Spanish student did not take Spanish in the fall semester).

Another option is to push students to discuss what types of functions are being modeled and to justify how they know their predictions are correct. Consider designating one student as the recorder to list ideas on the board. This list serves as a reference point for the classroom community to see the ideas that are elicited.

One way to transition from this conversation is to offer a brief summary of the ideas generated and debated. If students have not yet reached a consensus, acknowledge this fact and reassure students that they will have more opportunities to discuss their ideas.

CONNECTING TABLE VALUES TO ALGEBRAIC STRUCTURE

Next, we suggest a teacher-led and student-supported investigation of arithmetic patterns that arise in the tabular representations. These patterns illuminate the arithmetic structure that underlies the linear and exponential relationships. Further, these patterns establish a link between recursively defined and explicit (or closed-form) functions for the two relationships.

Consider **table 2**. This version of the table leaves the outputs in exact form, without the rounding that was applied in **table 1**. Filling in the table one row at a time allows for this repeatable question: How do we produce the next entry in this table? Ask for specifics on the roles of the –20 and the 0.95 in each table. Students should reiterate the constant rate of change in the linear function and the constant growth (decay) rate in the exponential function. By keeping track of the number of iterations, students can simplify the computations, thus approaching the explicit function rules in each.

Probe students' thinking about the appearance of the linear function $500 - n(20)$. Many students are conditioned to think of the linear function specifically as $y = mx + b$. The equivalent alternative here challenges students to revisit such concepts as the commutativity of the operations of addition and multiplication.

Successful navigation of this activity helps students break out of their typical thinking routines, thus allowing for new connections to their sense

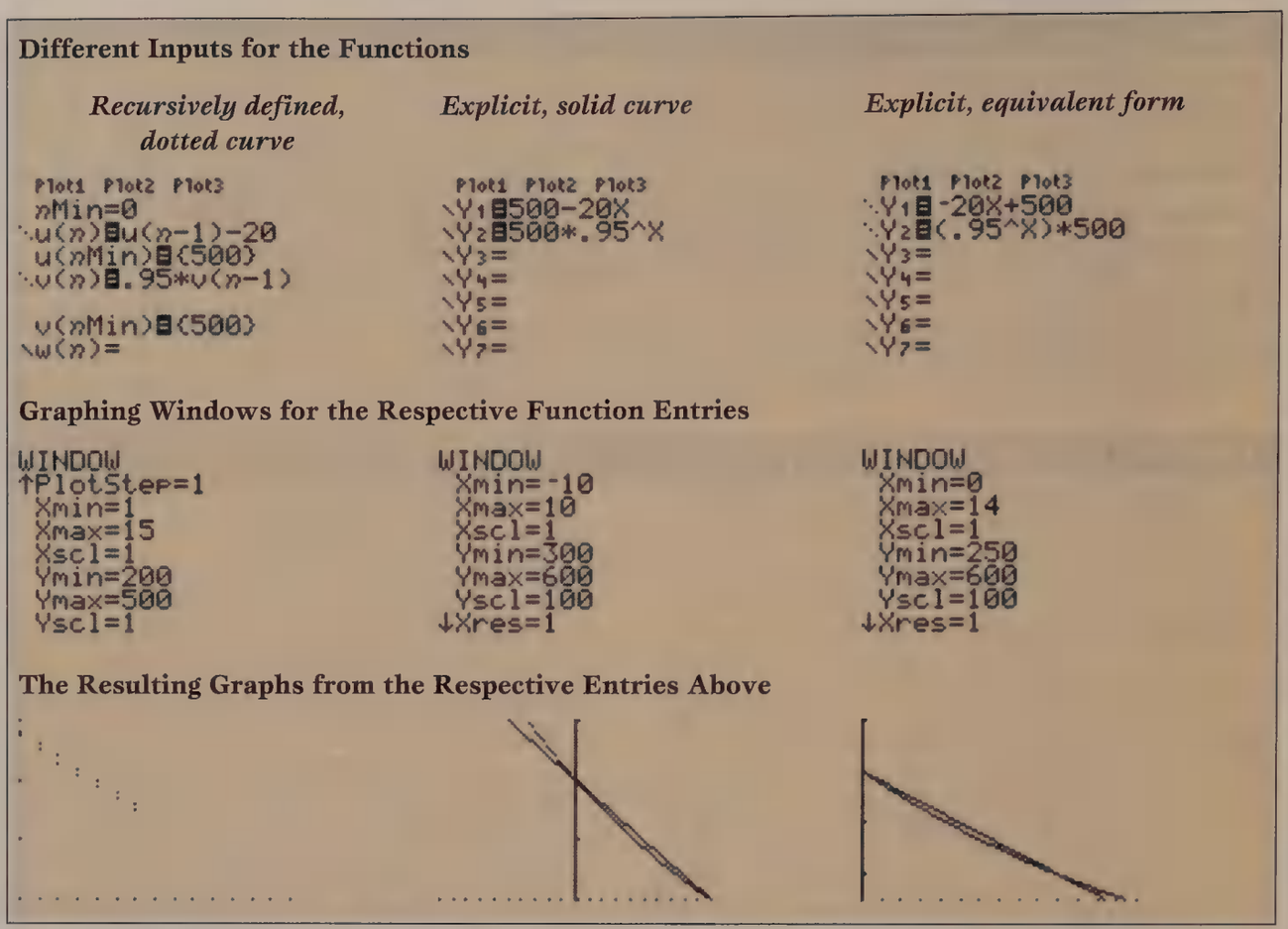


Fig. 3 Three possibilities of function and window configurations and the resulting graphs (each configuration is presented vertically) are shown.

of symbols. When these connections begin to take form and solidify in their minds, students will increase their capacity to think flexibly in new ways.

FROM THE ALGEBRAIC TO THE GRAPHICAL: CONNECTING REPRESENTATIONS

At this point, students have both recursive and functional rules for the two models in the Spanish word task. Some students may not have had an opportunity to explore the graphical patterns, so give them some time to regroup and produce some graphs of the data from the two models.

Students used graphing calculators to generate the images shown in **figure 3** (each configuration is presented vertically). Each option—which mode to use, how to set the viewing window, which type of line to plot (dotted vs. connected)—is a possible topic for provoking or sustaining discussion. Are the different equation representations equivalent? Are all these graphs correct? What viewing windows seem most appropriate for this task?

For the Spanish word lesson, we suggest a second round of whole-group conversation that revisits the concept of rate of change. Have students draw their graphs on the board or display their calculators or hand-sketched graphs on a document camera, if one is available. Select more than one group of students to display their work at the same time. Encourage students to share their thoughts publicly and to connect the formulas and graphs back to the tabular representations and the problem statement.

Challenge students to understand how different perspectives can be valid under different circumstances, an approach that promotes the iterative process of mathematical modeling. For example, consider the domain for these functions. In a general sense, no restriction exists that disallows negative inputs.

Also, encourage students to consider the difference between discrete and continuous models. Do they both match the original table values? Does it make sense to connect points on the graph or to look at inputs between the whole numbers on the table? What does it mean when the input is a positive nonwhole number such as $x = 1.2$ or $x = 25/7$?

Some students may also note that in model 1 the rate of decrease is a constant –20 after each week passes. This arithmetic pattern suggests a linear equation and confirms the tabular patterns established earlier. In model 2, the rate of change is not constant. Look for students who observe that, as time passes, the function decreases by less and less, showing signs of a leveling-off pattern. Pos-

sible prompts include such questions as these: What feature in the description of model 2 contributes to this leveling off? What characteristics in the tabular or algebraic representations lead to such leveling-off effect in the graphical representation?

DEEPENING UNDERSTANDING OF GRAPHICAL STRUCTURE

The electronic resource for the Spanish word task (www.nctm.org/geowords/) uses the functionality of GeoGebra, a free downloadable interactive software that supports both graphical and algebraic inputs. This website may be accessed by a single user in a whole-group demonstration and discussion or by students working in pairs or small groups. The website includes sketches designed as demonstrations, and students can use the sliders to explore the effects of changing one or more parameters (such as the values of an input, x_1, and change in inputs, Δx).

The first set of sketches (see **fig. 4a**) allows students to investigate within the constraints of the Spanish word task, where the slope and y-intercept of model 1 are fixed and the rate of decay and starting value of model 2 are fixed. The second set of sketches (see **fig. 4b**) allows students to go beyond the Spanish word task and alter the values of m and b in $y = mx + b$ or a and b in $y = a(b)^x$, respectively. As such, the second set of sketches is useful as an extension of the Spanish word task to other linear and exponential relationships.

Exploration of this interactive website can take between ten and twenty minutes. Teachers can encourage students to consider more generalized cases, as when the inputs are not merely positive integers. Will Δy remain constant for model 1 if $\Delta x = 2$, 3.5, or any values for any given input, x_1? Are all these constants the same or different from m? If different, are they related to m in any particular way? Have students work in groups to test different inputs of x_1 with respect to different choices for Δx and post their results in a table on the board for all to see. Students should notice that Δy indeed remains constant for any selected Δx. If the slope is constant and the slope is measured by the ratio of these values, then Δy must remain the same if Δx does not change. This result relates to the table in that the first differences are always –20 for model 1 (the slope of the graph).

Do similar structures exist for model 2 (and exponential graphs in general)? Because students may change the values for x_1 and Δx, the pattern

Technology Tips

Which technologies enrich your teaching? Which motivate your students?

Do you have a great use of technology in your classroom? If so, the editors of Technology Tips want to hear from you. Send us a manuscript with the step-by-step directions needed for using the technology with students. What are the potential pitfalls? How does the technology help students understand mathematics better?

Perhaps you have some reflections on technology that you would like to share. What are your recommendations for productive and innovative ways to use technology to improve student learning? How does technology lead students to explore and deepen their conceptual understanding, engage in mathematical investigations, model mathematical situations, and communicate conclusions?

Or perhaps you are expert in surfing the Net for mathematical websites to augment your lessons. Drop the editors a quick note about an exceptional website that leads you or your students to further understand or simply play with mathematics. Make the next Surfing Note *your* favorite website.

You may submit your completed manuscript for review by accessing **mt.msubmit.net.** Indicate that the manuscript is being submitted for the Technology Tips department. Be sure to enter the department name in the Department/Calls field. No author identification should appear in the text of the manuscript. Additional guidelines for preparation of manuscripts can be found at **www.nctm.org/publications/content.aspx?id=10452.**

CALL FOR MANUSCRIPTS

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

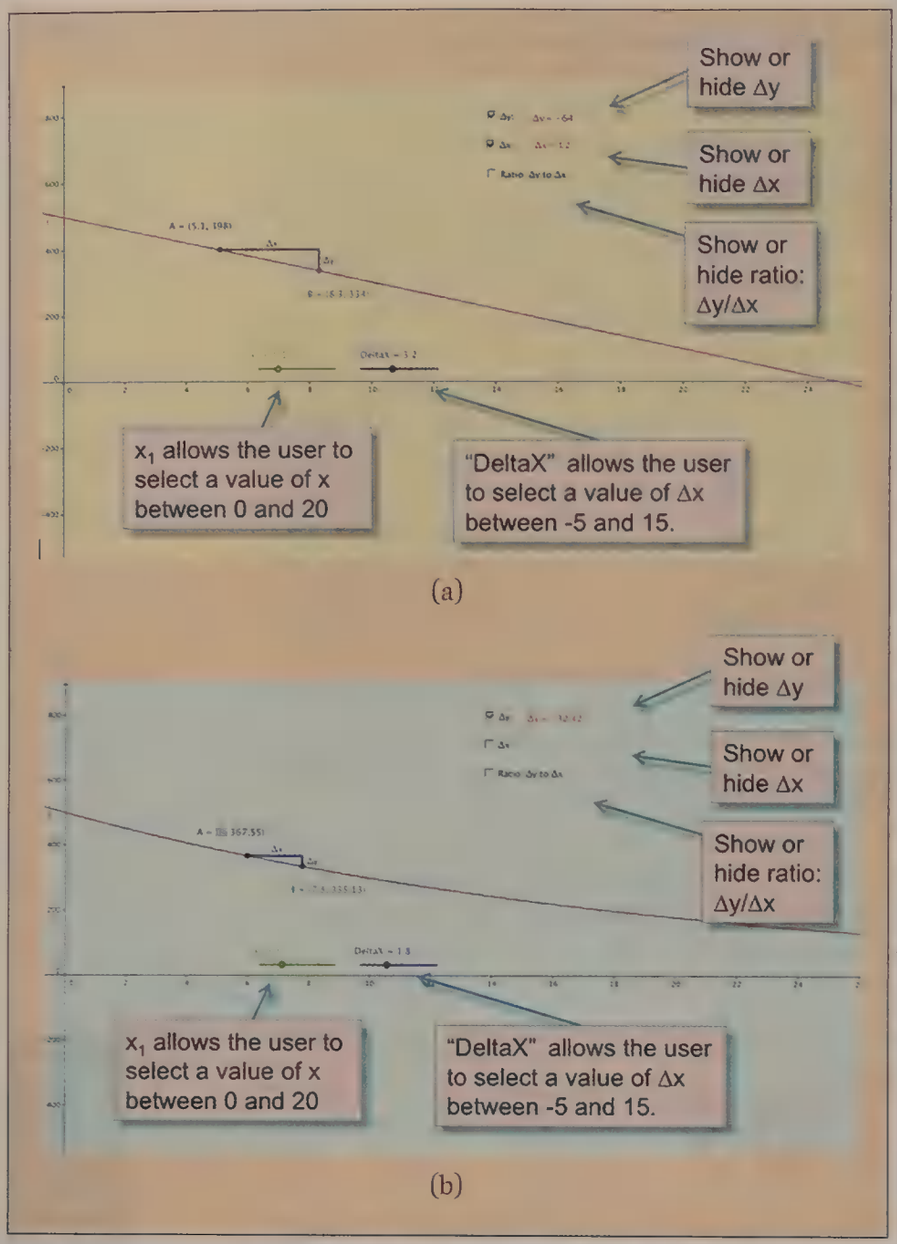

Show or hide Δy
Show or hide Δx
Show or hide ratio: Δy/Δx
x1 allows the user to select a value of x between 0 and 20
"DeltaX" allows the user to select a value of Δx between -5 and 15.
(a)
Show or hide Δy
Show or hide Δx
Show or hide ratio: Δy/Δx
x1 allows the user to select a value of x between 0 and 20
"DeltaX" allows the user to select a value of Δx between -5 and 15.
(b)

Fig. 4 Various options are available on the Web page for both linear (**a**) and exponential (**b**) models.

may not be immediately apparent if they do not investigate this question systematically. If students do not discern the pattern, prompt them to perform an experiment similar to the one performed for model 1. Have them test different inputs of x_1 with respect to different choices for Δx. Students should notice that Δy no longer remains constant for any selected Δx. When the ratio is viewable in real time, students can see Δy decrease as x_1 increases and as Δx remains fixed.

Again, push students to explain what is happening. Students should relate the curved nature of the graph to the effect of taking 95 percent of smaller and smaller numbers (the leveling-off pattern). This answers the question posed during the tabular exploration: Model 2 shows a decrease of 25 words between weeks 0 and 1 but a decrease of about 24 words between weeks 1 and 2 because 95 percent of 475 is less than 95 percent of 500.

A key element to the Spanish word problem is that the answer to part (*b*) is not clear-cut. Students who select model 1 may contend that it seems reasonable that a person would forget (roughly) the same number of words each week. Other students may argue against model 2, claiming that a student would more likely forget *more* words each week as time passes, not fewer. In contrast, proponents of model 2 may conjecture that it is reasonable for students to forget a small percentage of Spanish words in their vocabulary each week. Further, they may state that some Spanish words are never forgotten—thus, model 1 approaches zero too quickly. Inevitably, neither model is a perfect answer, and students should grapple with this fact.

Such struggles challenge students to act as authentic problem solvers and to think in flexible ways. Students may consider using piecewise-defined functions, regardless of whether they have spent significant time studying this concept. This approach opens up new possibilities for tackling the problem and thinking in flexible ways, useful approaches to future real-life modeling tasks.

SUMMARIZING THE LESSON

To wrap up this lesson, the teacher may select one student to give a recap of model 1 and its strengths and weaknesses in addressing the loss of Spanish words in a person's vocabulary and a second student to do the same for model 2. Guide each student to talk about each representation: tabular, algebraic, and graphical. Alternatively, the teacher may offer a summary.

As an assignment, have students complete a journal entry or a short reflective paper in which they document and defend their answers to the Spanish word task. They should also include the ideas that challenged them the most and the ideas that seemed to help their understanding the most. Prompt students to be clear about areas in which they can now think in new ways. Students can be pushed to write at least 400 words in their reflections. Give them a few minutes during class to begin an outline or write their reflections and ask questions of their peers; the remainder can be assigned as homework. Students should be aware that justifying their thinking is critical.

WHAT STUDENTS GAIN

The Spanish word lesson illuminates the connections between linear and exponential patterns from the perspective of structure. In particular, students explore the concepts of the rate of change from three different representations: tabular, graphical, and algebraic. These types of observations and investigations strengthen students' abilities to distinguish between linear and exponential functions.

In addition, the task fosters the development of flexible student understanding of operations, mathematical symbols, and rate of change. Last, the lesson can be used as a framework for designing other lessons that prompt students to slow down their thinking, dig deep, and build on their structural fluency and ability to think flexibly.

REFERENCES

Coxford, Arthur F., James T. Fey, Christian R. Hirsch, Harold L. Schoen, Gail Burrill, Eric W. Hart, Brian A. Keller, Ann E. Watkins, Mary Jo Messenger, and Beth Ritsema. 2003. *Contemporary Mathematics in Context: A Unified Approach*. Core Plus Mathematics Project. Columbus, OH: Glencoe/McGraw-Hill.

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

Cuoco, Al, E. Paul Goldenberg, and June Mark. 2010. "Contemporary Curriculum Issues: Organizing a Curriculum around Mathematical Habits of Mind." *Mathematics Teacher* 103 (9): 682–88.

Lee, Lesley, and Viktor Freiman. 2006. "Developing Algebraic Thinking through Pattern Exploration." *Mathematics Teaching in the Middle School* 11 (9): 428–33.

Teuscher, Dawn, and Robert E. Reys. 2010. "Connecting Research to Teaching: Slope, Rate of Change, and Steepness: Do Students Understand These Concepts?" *Mathematics Teacher* 103 (7): 519–24.

[illegible] is an [illegible] sor [illegible] in [illegible] ties [illegible] a PhD [illegible] at [illegible] He is [illegible] in [illegible] and to [illegible]

NCTM Needs You!

Call for Nominations
2013 Board of Directors Election

Each year, NCTM's Board of Directors makes important decisions that set the direction for the Council and mathematics education. The Board needs a broad representation of NCTM membership to benefit its discussions, inquiries, and decisions. In 2013, at least one high school teacher must be elected to ensure the balanced representation required by the bylaws.

NCTM has among its members many talented, energetic individuals who are qualified to assume leadership roles in the Council. The Nominations and Elections Committee needs your help in identifying these members by nominating them for Board Director positions.

Do you know someone who would bring valuable experience, perspective, and judgment to the Board? Nominate that person!

Would you be willing to serve on the NCTM Board of Directors? *You* can have a great impact on mathematics education at the national level.

For qualifications and responsibilities of NCTM directors and officers, school incentives for Board service, or to submit the names of nominees to the Nominations and Election Committee by completing the online nominations form go to: **www.nctm.org/nominations**. The chair of the committee will invite the nominee to complete an application after the first of the year.

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Tasks That Make Connections through Represent

Joe Garofalo and Christine P. Trinter

By working through well-designed tasks, students can expand their thinking about mathematical ideas and their approaches to solving mathematical problems. They can come to see the value of looking at tasks from different perspectives and of using different representations. Following are four tasks that encourage high school students and preservice secondary school mathematics teachers to think flexibly about mathematical concepts and problems.

Use of these tasks supports NCTM's Representation Standard. In particular, NCTM (2000) argues, "In grades 9–12, students' knowledge and use of representations should expand in scope and complexity.... [Students] will need to be able to convert flexibly among these representations. Much of the power of mathematics comes from being able to view and operate on objects from different perspectives" (p. 361).

Students think resiliently about using the quadratic formula, analyzing factors graphically, finding the shortest distance between two points, and finding margin of error.

ations

When using these tasks, we often ask learners to respond to them in as many ways as they can. Solutions can be found without using technology, but working through some of the tasks with technology can facilitate more insight into the various representations; by reducing the time needed to generate representations, technology allows students more time for analysis. Appropriate use of these tasks challenges learners' intuitions while

VTOROUS/ISTOCKPHOTO

broadening and deepening their knowledge of mathematical concepts through the linking of multiple representations.

TASK 1: EXPLORING THE QUADRATIC FORMULA

Most algebra students can recite the quadratic formula. However, when challenged to explain the meaning of the various parts of the quadratic formula, even preservice teachers have difficulty describing what each variable represents and the relationship between them.

We ask students to derive the quadratic formula from the standard form of the quadratic equation: $ax^2 + bx + c = 0$. Many do so by completing the square, whereas others need a hint. After students use the usual "completing the square" strategy to derive the formula, we have them leave the last line as

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

and not combine the fractional algebraic expressions, as is traditionally done. In particular, we encourage students to think about what the expression $-b/2a$ represents in the solution and how it relates to the parabolic graph of a quadratic function.

After students have the opportunity to think about the solution in this way, most realize that the first part of the solution, $-b/2a$, represents the axis of symmetry. They also realize that the roots of the equation are the zeros of the graph, found at equal distances from this axis, i.e.,

$$\pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

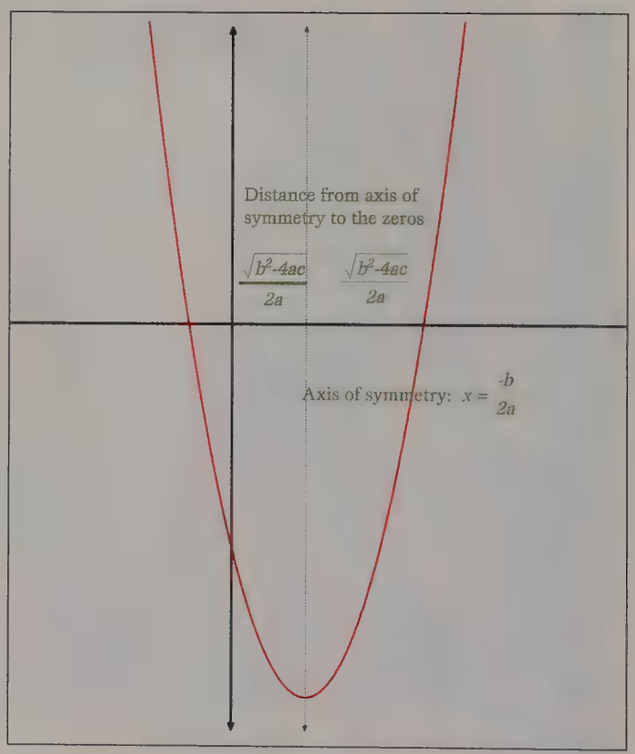


Fig. 1 Using the symmetric properties of the quadratic formula can lead to deeper understanding.

A graphical representation often helps students see the importance of its symmetric quality. The visual representation in **figure 1** shows that the distance from the axis of symmetry to each zero is

$$\pm\frac{\sqrt{b^2 - 4ac}}{2a} \text{ units}$$

when $b^2 - 4ac > 0$. Students usually then have no trouble explaining why there is only one solution when $b^2 - 4ac = 0$ and no real solutions when $b^2 - 4ac < 0$.

TASK 2: ANALYZING FACTORS GRAPHICALLY

When studying polynomials, high school students typically multiply binomials to get polynomials, factor polynomial expressions, and later graph polynomial functions. The following task helps students see connections among these.

Instead of asking students to multiply algebraic representations of binomial expressions to get a polynomial expression, we ask them to "multiply" the graphical representations of linear functions to approximate the graph of their product function. For example, **figure 2a** shows the graphs of $f_1(x) = x - 1$ in blue, $f_2(x) = x + 2$ in red, and $f_3(x) = x - 3$ in green. We ask students to analyze any "interesting or useful features of the graphs and regions" to determine, without multiplying the algebraic representations, the graph of the product function: $f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x)$.

Students are not accustomed to multiplying graphical representations, and some do not readily think of multiplying them point-wise. But most students will eventually recognize that when the value of one linear function equals zero, the value of the product function equals zero, and that when $x = 0$, the value of the product function equals $(2) \cdot (-1)(-3) = 6$. Students generally plot the four points resulting from that initial analysis $\{(-2, 0), (1, 0), (3, 0), (0, 6)\}$ to approximate the graph of the product function (see **fig. 2**).

Given some hints when needed, students notice that for all values of $x < -2$, all the values of the three functions are negative; thus, their product is negative, and the graph of the product function will be below the x-axis in that region. Students further reason that between $x = -2$ and $x = 1$, all the values of the two functions $f_1(x)$ and $f_3(x)$ are negative and that all the values of $f_2(x)$ are positive, so the graph

of the product function will be above the x-axis in that interval.

Similar reasoning leads students to determine that the graph of the product is again below the x-axis when $1 \le x \le 3$ and that it is above the x-axis when $x > 3$. With this information, most students draw reasonable approximations to the actual graph of the product function, although sometimes their local maxima and minima are not approximated well (see **fig. 2c**).

Most students and even some teachers who have completed this task state that they had not previously thought about function multiplication in this way. The analysis and reasoning used here helps students better connect algebraic, numerical, and graphical representations of functions. Further, the task can be a first step toward helping students get a sense that a function can be viewed as both a process and an object that can be manipulated (Sfard 1992).

TASK 3: UNDERSTANDING THE SHORTEST DISTANCE BETWEEN TWO POINTS

The third task convinces students and preservice teachers of the need to look at problems from multiple geometric perspectives:

> A fly and a spider are inside a room measuring 10 ft. × 10 ft. × 20. ft. (see **fig. 3**). The fly is centered on the back wall, 1 ft. below the ceiling, and the spider is centered on the front wall, 1 ft. above the floor. Find the shortest distance the spider must walk (along the walls, floor, or ceiling) to get to the fly. Show your method of solution.

The vast majority of students and preservice teachers answer this question by thinking of the room as a box, drawing a line like that in **figure 4a**, and then giving an answer of 30 feet. However, many who give this response add that the task seems "too easy." Some students then draw paths diagonally across the sides of the box. They construct several right triangles and use the Pythagorean theorem to find the lengths of these diagonals. As a result, they complicate the problem unnecessarily.

Most students open up only the box with the end walls being attached to the floor (see **fig. 4a**). Limiting themselves in this way leads to a "shortest" distance of 30 feet. Sometimes after a hint, students open up the box with the end walls being attached to another wall (see **fig. 4b**) or even the ceiling. Now they can see that shorter paths are possible. A physical model of the room is helpful for most students.

In this task, students ultimately use reasoning from another perspective to broaden their

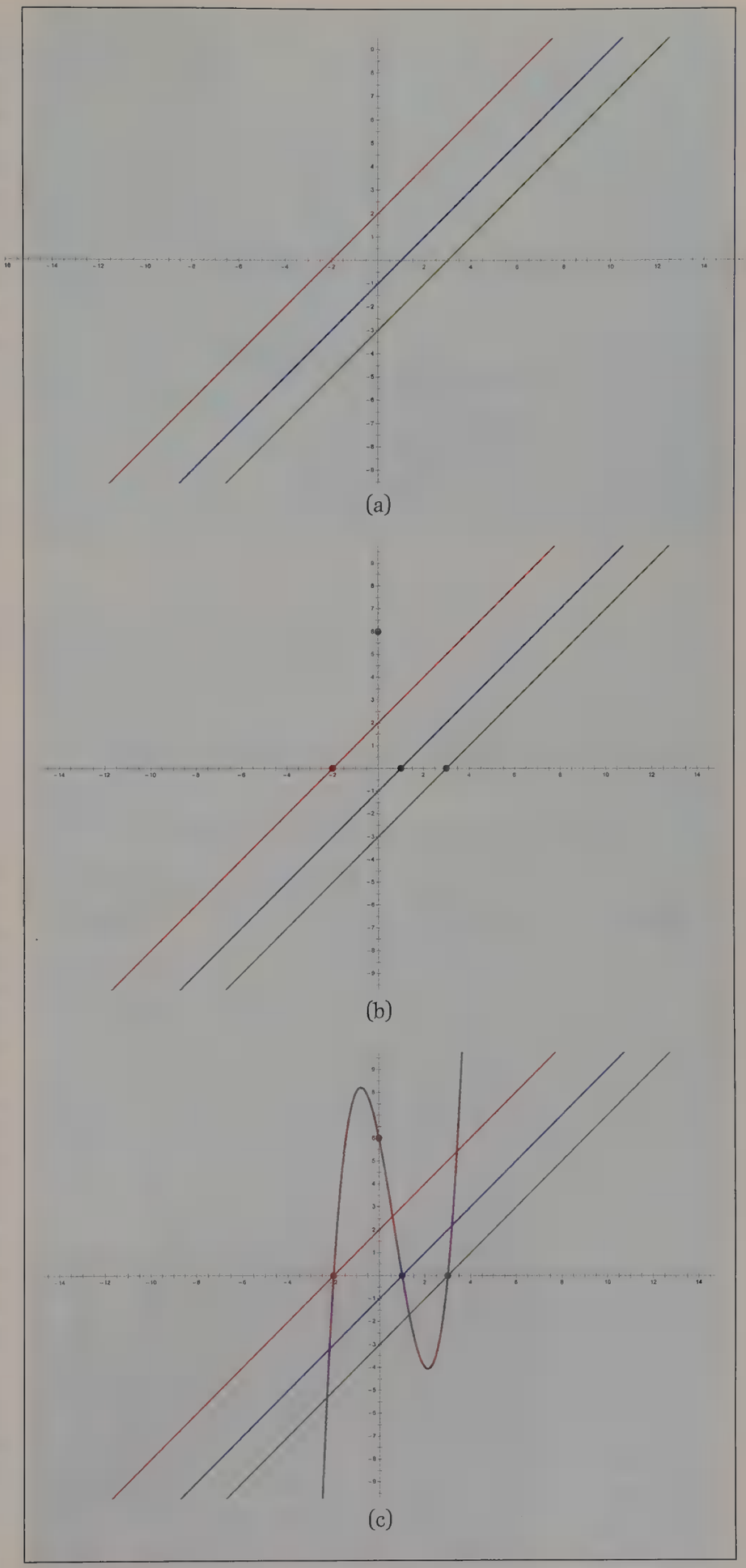


Fig. 2 The lines (**a**) can be used to determine the x- and y-intercepts (**b**) and then the graph of the cubic (**c**).

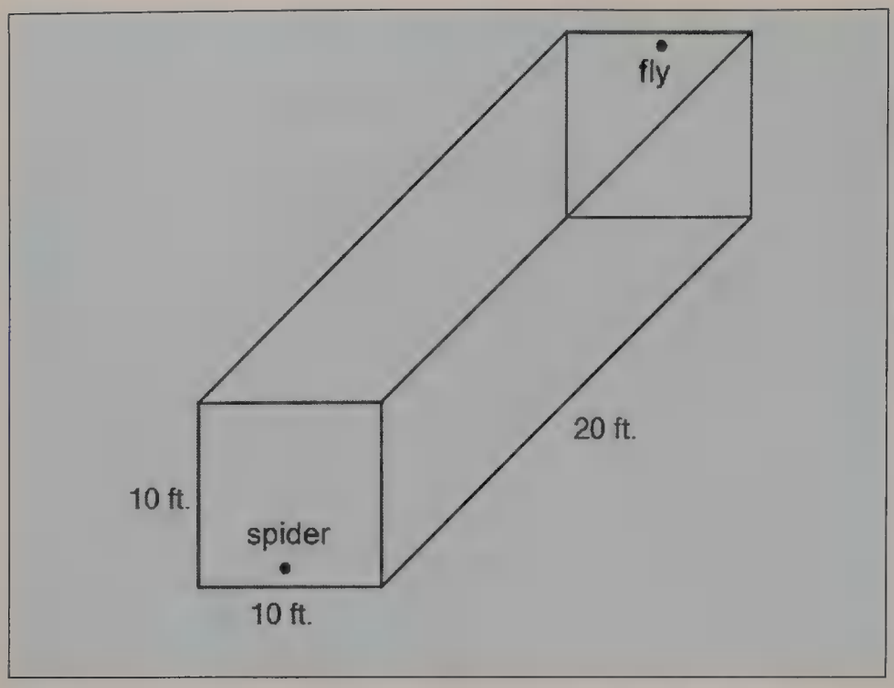


Fig. 3 What is the shortest path for the spider to follow to reach the fly?

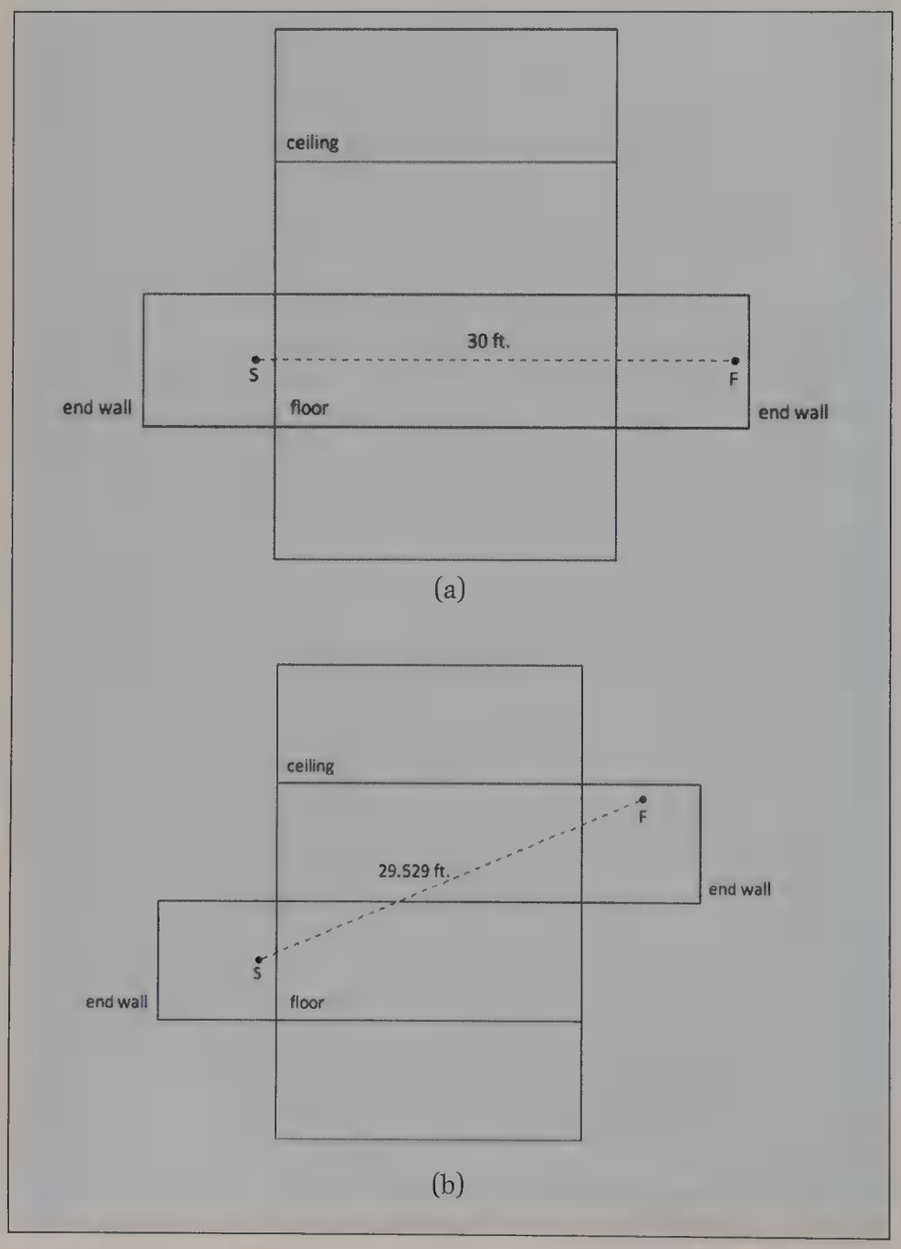


Fig. 4 The "shortest" path (**a**) can be made shorter (**b**) by thinking outside the "box."

range of strategies by manipulating two- and three-dimensional representations of the problem situation to help them find a solution. Engaging in this task shows students that problem solving often requires multiple attempts and allows them to think flexibly about types of problem-solving strategies.

TASK 4: UNDERSTANDING MARGIN OF ERROR

For the fourth task, we typically scour news media reports for results of a recent poll and then present students with these results and a related question. For example, we asked mathematics majors preparing to teach who had completed two courses in probability and statistics to explain the meaning of the margin of error in the poll summary given below. We also asked them how this margin of error might have been determined:

> A CBS poll released on May 22, 2011, reported that "a solid majority (77%) still say the nation's economy is in bad shape." The authors of the report stated that the poll was conducted "among 1,022 adults nationwide.... The error due to sampling for results based on the entire sample could be plus or minus three percentage points. The error for subgroups is higher."

Most of the students asked to explain what a margin of error of ±3% means responded that the true percentage of adults who think that the national economy is in bad shape is somewhere between 74% and 80%. This interpretation is no different from interpretations given by young students and adults with little or no formal knowledge of mathematics. In fact, an eighth-grade student told us, "It means 77% of the people they asked think that, but maybe for everyone, it is off a few percent." When we presented this interpretation to these mathematics majors, one responded, "I'm there with the middle school kid."

Over the years of giving this type of task to high school statistics students and preservice secondary school mathematics teachers, we have observed that most cannot readily explain how pollsters determine margin of error, nor could they explain why samples of this size (e.g., 1,000) are reasonable for a national poll. After some leading questions, many of these students ultimately recognize that this is an application of a one-proportion z-test, most likely with a 95% confidence interval—that is,

$$\left[\hat{p} \pm 1.96\sqrt{\frac{\hat{p}\left(1-\hat{p}\right)}{n}}\right].$$

We then suggest that students view the expression for the margin of error for this test not only as an algebraic formula, as is typically done, but also as a function of sample size. Then, by graphing and tracing the function

$$y_1(x) = 1.96\sqrt{\frac{(.23)(.77)}{x}}$$

for various sample sizes, students can verify the margin of error for the polls (e.g., for a sample of 1,022, with a proportion of .77, the margin of error is approximately ±2.6%). In addition, students can also easily determine the margin of error for different samples and hence answer the question, "Why does the report state that the error for subgroups is higher?" (See **fig. 5**.)

Further, by observing the shape of such graphs, students can see that increasing the sample size past 1,100 or 1,200 does not have much effect on the margin of error. In fact, tracing the graph for this particular poll shows that, for this proportion (.77), we will need to sample about 7,000 adults to have a margin of size ±1.0%. Students then use this diminishing return to explain why pollsters almost always sample around 1,100 adults.

Many other questions can be explored with such graphical representations. For example, we ask students to predict and demonstrate the position and shape of the graph of a standard error function for a 99% confidence interval; the shapes of the standard error function graphs when the proportions under consideration are close to 50% and close to 10%; and the graph of a standard error as a function of proportion for a given sample size. Viewing algebraic formulas as functions and comparing the graphs of these functions helps students gain more insight into polling and statistical inference.

Students who cannot answer these questions with conviction and sophistication, despite having been previously taught the necessary concepts and procedures, have difficulty connecting their study of inferential statistics to actual situations. One student remarked, "I can do the problems in the book, but I was confused by this one." Students need more experience applying what they are learning in school to situations outside their textbooks and their classrooms.

CONCLUSION

These four tasks involve mathematical formulas and concepts that are typically included in high school curricula. Moreover, they provide students opportunities to operate on and reason from different representations and ideas in unique ways. Students who solve these tasks and others like them are subsequently motivated to think more flexibly when solving other problems, to try alternative methods, to consider algebraic and geometric representations and solutions, and to evaluate their solution methods and intuitions.

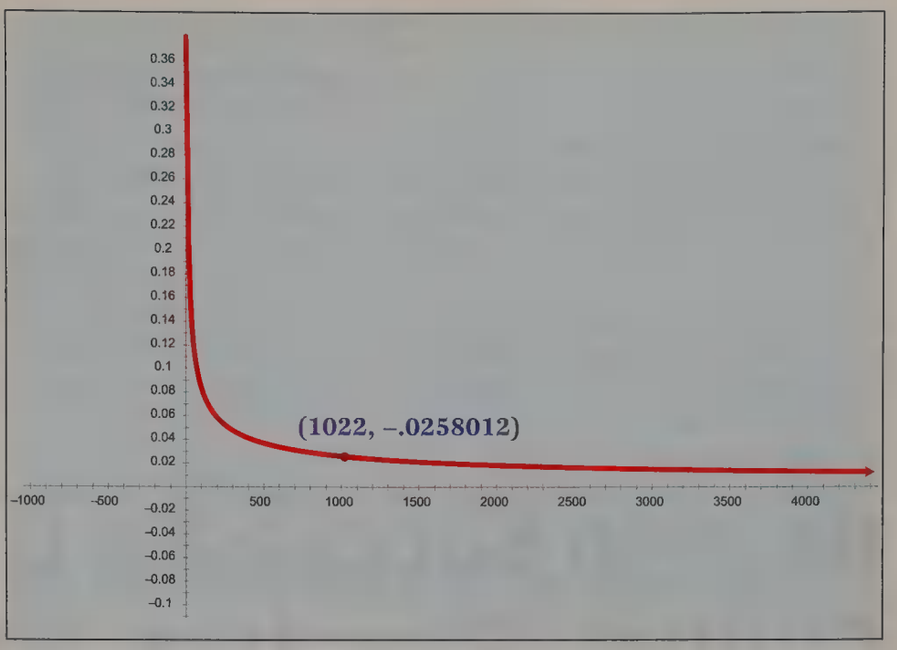


Fig. 5 Graphing the margin-of-error function allows students to understand why 1,100 is such an important number for pollsters.

These tasks substantiate NCTM's statement that "different representations support different ways of thinking about and manipulating mathematical objects" (NCTM 2000, p. 361).

REFERENCES

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, VA: NCTM.

Sfard, Anna. 1992. "Operational Origins of Mathematical Objects and the Quandary of Reification: The Case of Function." In *The Concept of Function: Aspects of Epistemology and Pedagogy*, edited by Guershon Harel and Ed Dubinsky, pp. 59–84. *MAA Notes*, vol. 25. Washington DC: Mathematical Association of America.

JOE GAROFALO, jg2e@virginia.edu, is codirector of the Center for Technology and Teacher Education at the University of Virginia in Charlottesville. His interests include mathematical problem solving, educational technology, and mathematics teacher education. CHRISTINE P. TRINTER, cpt5q@virginia.edu, is a research scientist at the University of Virginia. Her interests include educational technology, differentiating mathematics instruction, and mathematical problem solving.

Kasi C. Allen

Keys to Successful Group Work: Culture, Structure, Nurture

Connecting Research to Teaching appears in alternate issues of *Mathematics Teacher* and brings research insights and findings to the journal's readers. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#connecting.

Edited by ***Margaret Kinzel,*** *mkinzel@boisestate.edu*
Boise State University, Boise, ID

Laurie Cavey, *lauriecavey@boisestate.edu*
Boise State University, Boise, ID

The small-group approach changes the teacher's focus from being answer-oriented to being process-oriented.

—Weissglass (1990, p. 307)

WHY GROUP WORK? A BRIEF HISTORY

When I began teaching mathematics in the 1980s, few of my colleagues used group work, and those who did had minimal success. At the time, most high school classrooms were furnished with individual desks organized in rows, where students sat and listened—taking notes, practicing problems, seldom having the opportunity to collaborate with their peers—as students had for generations. Decades later, rows are still common but not nearly as pervasive. With the publication of NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989) and the focus on equity that followed, interest in cooperative learning grew, particularly with efforts to "de-track" mathematics in the high school. Educators recognized group work as an innovative practice with strong potential for meeting the learning needs of all students in heterogeneous classrooms.

Throughout the 1990s, other factors coincided to stimulate interest in group work. Numerous researchers—from Cohen (1994) to the National Research Council (NRC 2000) added to the professional knowledge base supporting the use of learner-centered practices. Simultaneously, the National Science Foundation's investment in curriculum development yielded an array of research-based high school mathematics programs, such as the Interactive Mathematics Program (www.mathimp.org/) and Core Plus Mathematics (www.wmich.edu/cpmp/), encouraging small groups as an instructional strategy.

Across multiple decades and disciplines, research suggests that group work at the high school level affords particular learning benefits for students, mathematically as well as socially (Cohen 1994; Davidson 1990; Johnson and Johnson 1992; Willis 2010; Watanabe 2012). Mathematically speaking, students who experience well-implemented group work—

- have more opportunities to articulate their thinking;
- exhibit deeper understanding and retention of concepts;
- welcome the ideas of others and can incorporate them into their own strategies; and

- feel less isolated and anxious about mathematics.

With respect to social outcomes, students learn to—

- communicate effectively, justifying their position through shared objective facts rather than emotional persuasion;
- work in a team environment, a skill increasingly needed in the workplace; and
- respect differences, particularly those relating to race and social position.

Despite the documented benefits of group work, a gap persists between research and practice. Too often, attempts to implement cooperative learning at the secondary school level do not reflect teachers' professional research-based knowledge. In my work with preservice teachers and their mentors, managing students in small groups remains one of the most common instructional challenges. I hear comments like these:

- "The students won't take [group work] seriously."
- "My strong students do all the work, and my weak students coast."
- "I think it's a great idea, but my students would rather work alone."
- "I have too many students to do group work."

Such comments reflect what research tells us: Successfully implementing group work does not happen without strategic preparation and instructional intent. Seating teenagers together in teams and giving them a common task may be a first step, but any positive results will be short-lived if teachers do not attend to critical details, particularly in the mathematics classroom.

Mathematics can serve as a uniquely valuable academic context for using group work. However, teachers also face mathematics-specific challenges with respect to issues of status. High school students have a strong sense of who does well in mathematics and who does not. These informal assessments of mathematics ability as well as perceived social standing operate as sources of status in the classroom. If a teacher does not plan to mediate such inequities in the classroom, group work can exacerbate the issues that it could potentially alleviate, such as student motivation or differentiated learning outcomes.

Years of following the research and refining my own practice have confirmed for me three key elements of group work in high school mathematics. If all three are present and teachers experience the results, they will incorporate group work into their pedagogy from then on. However, if even one element is missing, success remains elusive. The three key elements are (1) a classroom culture that supports collaboration and teamwork; (2) structure for groups and tasks; and (3) strategies that nurture the work throughout the year.

If a teacher does not plan to mediate such inequities in the classroom, group work can exacerbate the issues that it could potentially alleviate.

CULTURE

What the Research Says

Students need trusting relationships with one another as a foundation for building small groups that can learn together (Middleton and Jansen 2011). As a result, teachers must create opportunities for students to become better acquainted with one another, investing heavily throughout the year in activities that break down social barriers and develop trust among students. Within groups, as students come to know one another better, the process will transform them from a group of individuals, focused on self-interest, into a team, committed to ensuring that each person succeeds (Glasser 1986; Webb 1985).

Studies have also shown that high school students working in groups who experience making decisions on their own rather than being told what to do express a greater sense of control over and responsibility for their own learning (Sharan and Sharan 1992). In addition, they tend to view their teacher as a learner as well—leading to a significant shift in the power dynamic that generally characterizes the secondary school mathematics classroom (Cohen 1994).

In My Experience

Teenagers need time to talk about mathematics and to justify their thinking. Cooperative groups afford such an opportunity. However, students will not feel safe sharing their ideas openly in the mathematics classroom unless a supportive culture exists—one in which students and teachers exchange ideas, take risks, and are comfortable making mistakes. Such a classroom environment must be carefully crafted from the first day, and it begins with all students learning one another's names—a simple step, too often overlooked.

Teachers also foster a supportive culture by encouraging each class to view itself as a community of learners with its own identity. This process will require that teachers cede some control to students. In such classrooms, students adopt an attitude of "sink or swim together." As group members take more responsibility for one another's learning as well as their own, they increasingly view mathematics as something that they can do—rather than something that is done *to* them.

STRUCTURE

What the Research Says

Regardless of the task, to ensure engagement and equity, teachers should define roles for group work and distribute them so as to mediate potential imbalances (Cohen 1994). Roles help ensure that students with lower status—as a result of social characteristics or prior

mathematics experience—will be given an important role in the collective work. They can also help temper the behavior of students who tend to dominate interactions or take over (Shulman, Lotan, and Whitcomb 1998). For example, a highly verbal student might be assigned a role that requires more active listening, while a student with less mathematical confidence might launch the task.

Over the years, four classic roles have surfaced in the literature. These are also the standard roles for teams described in the College Preparatory Mathematics curriculum (see www.cpm.org):

- *Facilitator*—Initiates activity and checks for understanding throughout
- *Resource manager*—Manages supplies and seeks teacher assistance as needed
- *Task manager*—Listens actively and keeps entire team on track
- *Recorder* or *reporter*—Shares data with all and seeks consensus before reporting

Assigning roles provides teachers with a powerful tool for mediating inequities and promoting more equal participation within a group. However, students will still fail to collaborate if the task is not "group worthy," meaning that it is better accomplished by a group than by an individual.

Watanabe (2012) suggests six features of group-worthy tasks: a focus on a big idea; access by multiple abilities; open-endedness; interdependence; individual accountability; and assessment. In mathematics especially, tasks that demand careful consideration and discussion because there is not an obvious solution or answer work best. The key is to offer students what Willis (2010) refers to as an "achievable challenge" for the entire group. When group work defaults to students working alone and checking answers, chances are that the task at hand is too simple to be an appropriate group task. Ideally, group-worthy tasks bring students closer together because they need one another to navigate the complexity of the task successfully.

Over the past thirty years, mathematics educators have articulated a range of strategies and structures for transforming individual tasks into group tasks (Andrini 1996; Erickson 1995; Kushnir 2001). One such method involves using a clue-based structure (Erickson 1995), whereby each student takes responsibility for one clue, sharing the information orally. The group collectively makes sense of the clues to both define and solve the problem on a shared sheet of paper; see **figure 1** for an example.

Polygon 5 Your polygon has an area of 18 cm^2.	**Polygon 5** The longest segment that will fit inside your polygon is 6 cm long.
Polygon 5 Each side of your polygon has the same length. Each angle measure is the same too.	**Polygon 5** Your polygon has a perimeter of about 17 cm.
Source: Erickson (1995, p. 127)	

Fig. 1 This example of a clue-based problem used for group work in a geometry class concerns polygons.

Individual Exit Ticket Name: _____

1. Reflecting on your group's work today, write what you see as your group's greatest accomplishment and greatest challenge.
2. To what extent do you feel satisfied with the role you played as an individual? What will you do the same or differently as your group continues to work together?

Note: Your individual response is confidential. However, I will provide collective feedback to your group based on trends in all the exit tickets.

Fig. 2 An exit ticket can help assess how effectively a group functions.

In My Experience

Before assigning group roles, teachers will want to develop a vision for themselves of what group members will do in each role and clearly communicate these expectations to students. Whole-class discussions, including role playing, about what groups will look like if each person takes responsibility for his or her role in the collective work help ensure success. Classes that develop a strong sense of identity often create their own roles. Some examples that my classes have generated over the years follow:

- *Captain*—Ensures that everyone understands his or her role and participates
- *Timekeeper*—Tracks group work time and individual talk time
- *Skeptic*—Raises questions or doubts
- *Sound check*—Monitors volume level
- *Enforcer*—Keeps paper at the center of the table and keeps members in roles
- *Cheerleader*—Encourages positive attitude and persistence

Assigning roles fosters full participation in the group, but true cooperation necessitates engaging in a task that one person cannot readily accomplish alone. Problems of the Week (POWS), such as those appearing in the Interactive Mathematics Program (see www.mathimp.org/) or the Math Forum (see mathforum.org/pows/), are examples of readily available group-worthy tasks. Also, as a class develops into a community of learners, increasingly generating its own knowledge, students will begin to take interest in designing their own tasks. For teachers wanting to nudge their class in this direction, clue-based problems provide a simple format for students to initiate such work, often transforming a simple homework exercise into something much more engaging and complex.

NURTURE

What the Research Says

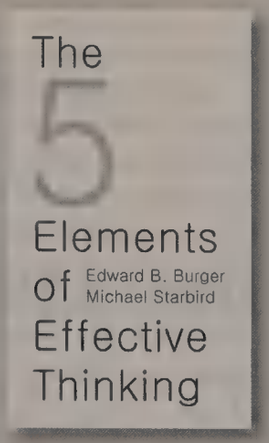
Even when teachers invest heavily in community building at the beginning of the year and organize well-designed group tasks in the months that follow, group work will deteriorate if teachers do not nurture the effort. Norms such as students relying on one another for answers to their questions, rather than turning immediately to the teacher as the source of knowledge, must be reinforced daily (Blumenfeld et al. 1996). In addition, teachers will want to invest in classroom activities that purposely attend to improvement, giving students opportunities to collectively debrief and evaluate their group experiences (Cohen 1994).

Throughout the year, small groups as well as the full class will benefit from team-building activities aimed at creating community. Varying tasks and activities, informed by regular student feedback, will also serve to keep group work fresh and exciting for students. To maintain motivation and interest, teachers need to make sure that students come to class curious about what the next hour will bring rather than readily able to predict what will transpire before the lesson has even begun (Willis 2010).

In My Experience

Classroom routines such as mathematics journals, exit tickets, and group quizzes can serve as powerful tools for nurturing the kind of individual responsibility and reflection that serve as the foundation for group work. Students can use their journals to write about their group experience, to test mathematical ideas before sharing them more broadly, and to communicate with the teacher about an issue that might be difficult to discuss face-to-face. Exit tickets—responses to short prompts completed on an index card or a half sheet of paper that check for understanding or poll for opinion—can be completed by individual students or by small groups. Group quizzes can focus on content or collaboration or both.

An example of an individual exit ticket aimed at assessing the effectiveness of the group is shown in **figure 2**. Ideally, students complete exit tickets



The 5 Elements of Effective Thinking

Edward B. Burger & Michael Starbird

"In this compact and remarkable book, two renowned professors share decades of teaching experience with anyone—from students to business people—seeking advice on how to improve skills and expand learning. It should be read, studied, and cherished—then reread."
—Fay Vincent, former commissioner of Major League Baseball and former president of Columbia Pictures

Cloth $19.95 978-0-691-15666-8

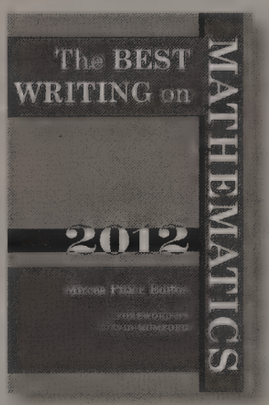
Guesstimation 2.0

Solving Today's Problems on the Back of a Napkin

Lawrence Weinstein

"*Guesstimation 2.0* is an entertaining read, with the added attraction that it can be consumed in small portions by opening it on almost any page. I can easily see having this book close by and returning to it again and again."
—Mark Levi, author of *Why Cats Land on Their Feet: And 76 Other Physical Paradoxes and Puzzles*

Paper with French folds $19.95 978-0-691-15080-2



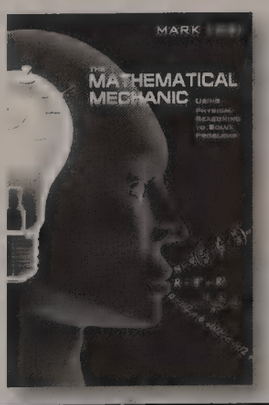
The Best Writing on Mathematics 2012

Edited by Mircea Pitici

Foreword by David Mumford

This annual anthology brings together the year's finest mathematics writing from around the world. Featuring promising new voices alongside some of the foremost names in the field, *The Best Writing on Mathematics 2012* makes available to a wide audience many articles not easily found anywhere else.

Paper $19.95 978-0-691-15655-2



The Mathematical Mechanic

Using Physical Reasoning to Solve Problems

Mark Levi

"*The Mathematical Mechanic* is an excellent display of creative, interdisciplinary problem-solving strategies. The author has explained complex concepts with simplicity, yet the mathematics is accurate."
—*Mathematics Teacher*

Paper $14.95 978-0-691-15456-5

PRINCETON UNIVERSITY PRESS

See our E-Books at press.princeton.edu

during the last few minutes of class and hand them directly to the teacher, who stands at the door as students leave the classroom.

When seeking to invest in the class as a whole, teachers may find that structured group activities that remix the students into new small groups for a short time or that encourage pairwise interaction help infuse new energy into the community-building process. Two strategies that I have found particularly useful are matching and lining up (Kushnir 2001). Matching activities involve individual cards (such as clue-based problem solving). However, in this case, the teacher predesigns groupings of two, three, or four cards that share a common theme or concept. In an algebra class, an activity could involve cards that display an equation, a table, a graph, or a verbal description. The teacher distributes a card to each student, and then students move about the room in search of the others whose cards "match" theirs. Once students have a match, the group sits down and justifies its decision on paper, demonstrating how each card is a different representation of the same relationship.

Similarly, lining-up activities provide an opportunity for pairwise, whole-class engagement. Again, each student receives a card. This time, however, the cards might have an expression to evaluate or an equation to solve. Each student works individually to obtain a result and then uses this value to determine his or her place in a numerical line-up of the class. Once they have taken a place in line, students verify their results with a neighbor and then make corrections and reshuffle as necessary. Once the class believes that it has an accurate line-up, the teacher walks down the line, calling on a few students to confirm that they are in the right order, or simply collects the cards in order, confirming as she or he goes.

GROUP WORK IS WORTH THE EFFORT

For a high school mathematics teacher, perhaps nothing is more satisfying than a classroom full of students who want to be there and who come to class ready to engage in compelling work. Well-implemented group work serves as a cornerstone of such a community of learners. When students learn to see each person in the classroom as an important resource, everyone benefits. High-achieving students learn to place less value on quickly finding the correct answers, whereas struggling students experience less isolation and gain more confidence. All students have the opportunity to expand their own thinking by embracing ideas and strategies beyond their own.

To teachers who have struggled with group work in the past: I urge you try again. If you have never tried, begin. Your students will thank you and then astound you with what they can do. Your teaching will never be the same.

REFERENCES

Andrini, Beth. 1996. *Cooperative Learning and Mathematics: A Multi-Structural Approach.* San Clemente, CA: Kagan.

Blumenfeld, Phyllis C., Ronald W. Marx, Elliott Soloway, and Joseph Krajcik. 1996. "Learning with Peers: From Small Group Cooperation to Collaborative Communities." *Educational Researcher* 25 (8): 37–40.

Cohen, Elizabeth. 1994. *Designing Group Work: Strategies for the Heterogenous Classroom.* 2nd ed. New York: Teachers College Press.

Davidson, Neil, ed. 1990. *Cooperative Learning in the Mathematics: A Handbook for Teachers.* Menlo Park, CA: Addison-Wesley.

Erickson, Timothy. 1995. *United We Solve: 116 Problems for Groups, Grade 5–10.* Oakland, CA: EEPS Media.

Glasser, William. 1986. *Control Theory in the Classroom.* New York: Harper and Row.

Kushnir, Dina. 2001. *Cooperative Learning and Mathematics: High School Activities.* San Clemente, CA: Kagan.

Johnson, David W., and Roger T. Johnson. 1992. *Learning Together and Alone: Cooperative, Competitive, and Individualistic Learning.* 3rd ed. Englewood Cliffs, NJ: Prentice-Hall.

Middleton, James A., and Amanda Jansen. 2011. *Motivation Matters and Interest Counts: Fostering Engagement in Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 1989. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

National Research Council (NRC). 2000. *How People Learn: Brain, Mind, Experience, and School: Expanded Edition.* Washington, DC: National Academies Press.

Sharan, Yael, and Shlomo Sharan. 1992. *Expanding Cooperative Learning through Group Investigation.* New York: Teachers College Press.

Shulman, Judith, Rachel Lotan, and Jennifer Whitcomb, eds. 1998. *Group Work in Diverse Classrooms: A Casebook for Educators.* New York: Teachers College Press.

Watanabe, Maika. 2012. *"Heterogenius" Classrooms: Detracking Math and Science: A Look at Groupwork in Action.* New York: Teachers College.

Webb, Noreen M. 1985. "Student Interaction and Learning in Small Groups: A Research Summary." In *Learning to Cooperate, Cooperating to Learn*, edited by Robert Slavin, Shlomo Sharan, Spencer Kagan, Rachel Hertz Lazarowitz, Clark Webb, and Richard Schmuck, pp. 147–72. New York: Plenum Press.

Weissglass, Julian. 1990. "Cooperative Learning Using a Small-Group Laboratory Approach." In *Cooperative Learning in Mathematics: A Handbook for Teachers*, edited by Neil Davidson, pp. 295–334. Menlo Park, CA: Addison-Wesley.

Willis, Judy. 2010. *Learning to Love Math: Teaching Strategies That Change Student Attitudes and Get Results.* Alexandria, VA: Association for Supervision and Curriculum Development.

KASI C. ALLEN, based in Kennewick, is assistant professor of mathematics education in the Graduate School of Education and Counseling at Lewis and Clark College in Portland, Oregon. She is interested in issues of mathematics equity and strategies for mathematically empowering students across the grade levels.

Martin Griffiths

Close Encounters with Stirling Numbers of the Second Kind

Mathematicians find it highly satisfying to recognize links between mathematical scenarios that initially might seem to be totally unrelated. This article grew out of my own experience of such an event. I had given one of my classes a problem associated with the factorization of integers, and then, a few days later, I gave another class a question on probability. I did not notice any connection between these two questions at the time, but on reflection I found that they were related in a subtle but profound way.

The ideas explored to demonstrate this link provide a rich source of material for students and teachers alike and present the opportunity to introduce students to recurrence relations, ordered and unordered partitions, sums of random variables, and the multiplicative structure of natural numbers. Although the ideas developed here are relatively sophisticated, both mathematical encounters occurred as a result of generalizing some fairly straightforward mathematics that took place in ordinary school classrooms.

ENCOUNTER 1

Stirling numbers of the second kind crop up in particular combinatorial situations. Our first encounter with these numbers arose from a task on factor sequences that I had given a year 8 class (equivalent to grade 7 in the United States), although the extended investigation presented here would be suitable for high school students.

For $n \geq 2$, we call the finite sequence $x_1, x_2, \ldots, x_k$ of integers of length $k \geq 2$ a *factor sequence* of n if the following conditions are satisfied:

(i) $x_1 = 1$ and $x_k = n$

(ii) $x_{m+1} > x_m$, $m = 1, 2, \ldots, k - 1$

(iii) $x_m \mid x_{m+1}$, $m = 1, 2, \ldots, k - 1$

For example, (1, 4, 24, 120) is a factor sequence of 120 of length 4.

To help students understand the multiplicative structure of integers, I asked them to find all possible factor sequences for numbers such as 6, 70, and 210 whose prime factorizations consist of distinct primes. The three factor sequences of 6 are (1, 6), (1, 2, 6), and (1, 3, 6), whereas the thirteen factor sequences of 70 are as follows

(1, 70), (1, 2, 70), (1, 5, 70), (1, 7, 70), (1, 10, 70), (1, 14, 70), (1, 35, 70), (1, 2, 10, 70), (1, 2, 14, 70), (1, 5, 10, 70), (1, 5, 35, 70), (1, 7, 14, 70), (1, 7, 35, 70)

Here we consider the more challenging problem of finding the total number of distinct factor sequences of n when n is the product of m distinct primes, $n = p_1 p_2 \ldots p_m$. Note that any two integers n_1 and n_2 that are the product of m distinct primes will have the same number of factor sequences by the fundamental theorem of arithmetic, which guarantees the uniqueness of the decomposition into primes up to permutation of the factors. Note that $15 = 3 \cdot 5$ has the same number of factor sequences as $6 = 2 \cdot 3$; that $231 = 3 \cdot 7 \cdot 11$ has the same number as $30 = 2 \cdot 3 \cdot 5$; and so on. The upshot is that if we let f_m denote the number of factor sequences of n, then f_m is well defined. For the sake of simplicity, therefore, we may assume

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

Edited by ***Maurice Burke****, Maurice.Burke@utsa.edu*
University of Texas at San Antonio

J. Kevin Colligan*, jkcolligan@verizon.net*
RABA Center of SRA International, Columbia, MD

Maria Fung*, mfung@worcester.edu*
Worcester State College, Worcester, MA

Jeffrey J. Wanko*, wankojj@muohio.edu*
Miami University, Oxford, OH

that n is the product of the first m prime numbers. From the examples given, we already know that $f_2 = 3$ and $f_3 = 13$.

Factor sequences are linked to partitions of the set A comprising the m prime factors of n. To *partition* set A is to split it up into subsets in such a way that the subsets collectively contain all of set A's elements but no two of them contain the same element. Suppose, for example, that $n = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ and, hence, that A = {2, 3, 5, 7, 11}. Then both {{2, 7}, {3, 5, 11}} and {{3}, {2, 7}, {5, 11}} are partitions of A, whereas {{2, 5, 7}, {3, 7, 11}} and {{3}, {2, 11},{7}} are not.

We highlight two key points. First, there is a one-to-one correspondence between subsets of A and the factors of n that are greater than 1. For example, the subset of A given by {2, 3, 7} corresponds to the factor of n given by $42 = 2 \cdot 3 \cdot 7$; conversely, the factor 42 gives rise to the subset {2, 3, 7}.

Second, given some partition of A, a particular ordering of the resultant subsets will correspond to a particular factor sequence of n. So if we start with {{3}, {2, 7}, {5, 11}} and order its elements as ({2, 7}, {3}, {5, 11}), this approach leads to the factor sequence (1, 14, 42, 2310) of length 4 since $2 \cdot 7 = 14$, $3 \cdot 14 = 42$, and $5 \cdot 11 \cdot 42 = 2310$. To calculate f_m, therefore, we just need to enumerate the number of ordered partitions of A.

This is a good time to introduce Stirling numbers of the second kind—formally, $S(m, k)$. They give the number of ways of partitioning m distinct objects into k nonempty parts (see Cameron [1994]; Sloane [2012], sequence A008277). For our purposes, the set of m distinct objects comprises the first m primes. To illustrate this idea, here are all the possible ways in which {2, 3, 5, 7} can be partitioned into three parts:

{{2},{3},{5, 7}}; {{2},{5},{3, 7}}; {{2},{7}, {3, 5}};
{{3}, {5}, {2, 7}}; {{3}, {7}, {2, 5}}; {{5}, {7}, {2, 3}}

This list tells us that $S(4, 3) = 6$. As may be seen in Cameron (1994), these numbers obey the recurrence relation

$$S(m, k) = kS(m - 1, k) + S(m - 1, k - 1) \qquad (1)$$

where $S(m, k)$ is defined to be zero unless either $1 \leq k \leq m$ or $m = k = 0$ (with $S(0, 0) = 1$ by definition). Tabulated values for $S(m, k)$ with $m, k \leq 10$ are given in the Wikipedia entry on Stirling numbers of the second kind (2012). We also note here that

$$S(m, 1) = 1$$

and

$$S(m+1, m) = \frac{m(m+1)}{2} \text{ for } m \geq 1. \qquad (2)$$

Now let's consider {{2}, {3}, {5, 7}}, the first of the partitions of {2, 3, 5, 7} into three parts (as given above). There are $3! = 1 \cdot 2 \cdot 3 = 6$ ways of ordering these three parts, and hence this particular partition gives rise to 3! distinct factor sequences of $n = 2 \cdot 3 \cdot 5 \cdot 7 = 210$, which are all of length 4. This reasoning also applies to each of the remaining five partitions of {2, 3, 5, 7} into three parts. Therefore, the number of factor sequences of 210 of length 4 is $3!S(4, 3) = 6 \cdot 6 = 36$.

Continuing in this manner, we can calculate f_4. Note that $S(4, 1) = 1$, $S(4, 2) = 7$, and $S(4, 4) = 1$ correspond to the partitions of {2, 3, 5, 7} given by

{{2, 3, 5, 7}};
{{2}, {3, 5, 7}}; {{3}, {2, 5, 7}}; {{5}, {2, 3, 7}}; {{7},{2, 3, 5}};
{{2, 3}, {5, 7}}; {{2, 5}, {3, 7}}; {{2, 7}, {3, 5}};
{{2}, {3}, {5}, {7}},

leading to factor sequences of lengths 2, 3, and 5. Then $f_4 = 1!S(4, 1) + 2!S(4, 2) + 3!S(4, 3) + 4!S(4, 4) = 75$.

We now derive a formula for f_m. By definition, there are $S(m, k)$ ways of partitioning m primes into k nonempty subsets. Thus, since k subsets may be ordered in $k!$ ways, it follows that there are $k!S(m, k)$ factor sequences for n of length $k + 1$. Summing over all possible values of k then gives us

$$f_m = \sum_{k=1}^{m} k!S(m, k).$$

ENCOUNTER 2

The origins of the second encounter lie in a question I had asked one of my lower sixth classes (equivalent to grade 11). I asked students to calculate the probability that exactly three distinct numbers occur when a fair six-sided die is rolled five times. For example, both 4, 1, 4, 4, 5 and 3, 3, 5, 2, 5 correspond to sequences of length 5 containing exactly three distinct numbers, whereas 2, 4, 5, 3, 4 does not.

While students were working on this task, I pondered a more general problem. Rather than restricting ourselves to the six possible outcomes on a normal die, suppose that we constructed a spinner from a regular polygon with k edges (numbered 1 to k). How might we calculate the probability $P(k, m, n)$ so that exactly n distinct numbers appear when the spinner is spun m times, assuming that each of the k outcomes is equally likely? This problem too turns out to have an interesting connection with Stirling numbers of the second kind.

We may regard the outcome of each of the m spins as either N or O where N represents a new number and O an old number (i.e., numbers that have and have not already occurred, respectively). For example, the sequence of outcomes 7, 3, 5, 2, 3, 4, 7, 3, 1, 4 obtained by spinning a fair eight-edged spinner 10 times would result in the sequence NNNNONOONO. The leftmost four Ns correspond to the initial four numbers—7, 3, 5, 2—whereas the first appearance of an O indicates that the fifth term of the sequence, 3, had already occurred before. The sixth number, 4, had not appeared before—hence, the N. The remaining terms in the sequence of Ns and Os are obtained similarly. A sequence of spins produces exactly n distinct numbers if and only if when it is converted to a sequence of Ns and Os, the number of Ns is equal to n.

To obtain the answer to our general problem, we simply sum the probabilities corresponding to all possible distinct sequences of outcomes containing n Ns and $m - n$ Os (noting that N is always the first outcome and that such sequences are not, in general, equally likely). To do so, we use a probability distribution known as the *geometric distribution*, as described by Grimmett and Stirzaker (2001).

Suppose that we have a biased coin for which the probability of obtaining a head is p and that we keep tossing the coin until a head appears. What is the probability that exactly r tosses are required? If it takes r tosses to obtain the first head, then a tail must appear on each of the first $r - 1$ tosses and a head on toss r. Thus, with the random variable X representing the number of tosses required, we have

$$P(X = r) = p(1-p)^{r-1}.$$

In this case, X is said to follow a geometric distribution with parameter p. If, for example, $p = 1/3$, then

$$P(x = 4) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{4-1} = \frac{8}{81}.$$

Returning to the problem, let $X_1 = 1$. Then, for $j \geq 2$, define X_j as the number of spins taken from immediately after the $(j - 1)$th distinct number had occurred to the point at which the jth distinct number first appears. By comparing this situation with that in the previous paragraph, we see that X_j follows a geometric distribution with parameter $(j - 1)/k$. Hence,

$$P(X_j = r) = \frac{k-j+1}{k} \cdot \left(\frac{j-1}{k}\right)^{r-1}$$

for $r \geq 1$.

Now we'll demonstrate how to calculate

Connecting Research to Teaching

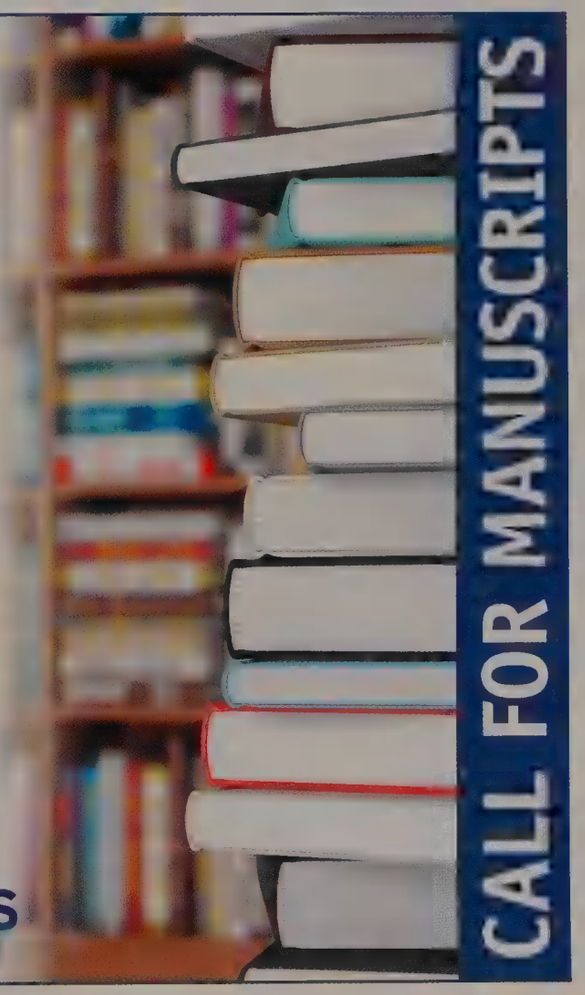
The Connecting Research to Teaching department of *MT* extends an invitation to readers to contribute to the department. CRT focuses on making connections between mathematics education research and classroom teaching. Manuscripts typically follow one of two formats:

- Highlighting how the author's work relates to secondary school classrooms
- Presenting an overview of research in a particular area to reveal implications for teaching

Articles published in this department are eligible for NCTM's Linking Research and Practice Outstanding Publication Award; for more about this new award, go to **http://www.nctm.org/news/content.aspx?id=31248**.

Manuscripts should include the research context and should be written with the classroom teacher as reader in mind. A good way to focus your material is to identify one clear message for teachers. Sharing student work where appropriate is encouraged. Manuscripts must be 2500 words or less.

Submission guidelines can be found at **http://www.nctm.org/publications/content.aspx?id=22602**. If you have questions about whether a particular idea would be a good fit for this department, contact Margaret Kinzel, mkinzel@boisestate.edu, or Laurie Cavey, lauriecavey@boisestate.edu.



NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

$P(8, 10, 3)$, the probability that exactly three distinct numbers appear when an eight-sided spinner is spun ten times. We have

$$P(8,\ 10,\ 3) = P\begin{pmatrix} (X_1 + X_2 + X_3 \le 10) \wedge \\ (X_1 + X_2 + \cdots + X_4 > 10) \end{pmatrix}$$

since, from the definition of X_j, the first inequality ensures that at least three distinct numbers appear, whereas the second ensures that the fourth distinct number does not appear until after the tenth roll. The probability $P(8, 10, 3)$ is obtained by summing over all such possibilities. For example, the probability corresponding to the three Ns occurring in the first three rolls (NNNOOOOOOO) is

$$\frac{8}{8}\cdot\frac{7}{8}\cdot\frac{6}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8},$$

whereas

$$\frac{8}{8}\cdot\frac{1}{8}\cdot\frac{1}{8}\cdot\frac{7}{8}\cdot\frac{2}{8}\cdot\frac{2}{8}\cdot\frac{2}{8}\cdot\frac{2}{8}\cdot\frac{6}{8}\cdot\frac{3}{8}$$

arises from NOONOOOONO. We need an efficient way of carrying out this summation.

Generalizing the above gives

$$P(k,\ m,\ n) = P\begin{pmatrix} (X_1 + X_2 + \cdots + X_n \le m) \wedge \\ (X_1 + X_2 + \cdots + X_{n+1} > m) \end{pmatrix}.$$

For example, the probability corresponding to k Ns occurring in the first k rolls is

$$\frac{k}{k}\cdot\frac{k-1}{k}\cdot\cdots\cdot\frac{k-n+1}{k}\cdot\frac{n}{k}\cdot\frac{n}{k}\cdot\cdots\cdot\frac{n}{k} = \frac{n^{m-n}(k)_n}{k^m},$$

whereas

$$\frac{k}{k}\cdot\frac{k-1}{k}\cdot\frac{2}{k}\cdot\frac{2}{k}\cdot\frac{k-2}{k}\cdot\frac{k-3}{k}\cdot\cdots\cdot\frac{k-n+1}{k}\cdot\frac{n}{k}\cdot\frac{n}{k}\cdot\cdots\cdot\frac{n}{k} = \frac{2^2 n^{m-n-2}(k)_n}{k^m}$$

results from the sequence NNOON ... NO ... O, with $(k)_n = k(k-1)\cdots(k-n+1)$. It can be seen that

$$P(k,\ m,\ n) = \frac{(k)_n}{k^m}\sum a_1 a_2 \ldots a_{m-n} \qquad (3)$$

where the sum is over all possible distinct products $a_1 a_2 \ldots a_{m-n}$ such that $(a_1, a_2, \ldots, a_{m-n})$ is a sequence of $m - n$ nondecreasing positive integers, with each a_i an element from $\{1, 2, 3, \ldots, n\}$. Note that $m \ge n$ and, when $m = n$, the expression $\sum a_1 a_2 \ldots a_{m-n}$ is equal to 1 by definition. We want to evaluate the sum $f(p, q) = \sum a_1 a_2 \ldots a_p$ over all possible distinct products of p nondecreasing positive integers, where a_i is an element from the set $\{1, 2, 3, \ldots, q\}$. For example,

$$f(2,\ 3) = 1\cdot1 + 1\cdot2 + 1\cdot3 + 2\cdot2 + 2\cdot3 + 3\cdot3 = 25.$$

As we now explain, f satisfies the following recurrence relation for $p, q \ge 2$:

$$f(p,\ q) = qf(p-1,\ q) + f(p,\ q-1) \qquad (4)$$

Multiplying each product of $p - 1$ integers given in $f(p - 1, q)$ by q gives rise to all the products in $f(p, q)$ containing at least one q, whereas $f(p, q - 1)$ contributes all the products in $f(p, q)$ not containing a q.

For example, to obtain $f(3, 3)$, we add

$$\begin{aligned} 3f(2,\ 3) &= 3(1\cdot1 + 1\cdot2 + 1\cdot3 + 2\cdot2 + 2\cdot3 + 3\cdot3) \\ &= 1\cdot1\cdot3 + 1\cdot2\cdot3 + 1\cdot3\cdot3 + \\ &\quad 2\cdot2\cdot3 + 2\cdot3\cdot3 + 3\cdot3\cdot3 \\ &= 75 \end{aligned}$$

to

$$\begin{aligned} f(3,\ 2) &= 1\cdot1\cdot1 + 1\cdot1\cdot2 + 1\cdot2\cdot2 + 2\cdot2\cdot2 \\ &= 15 \end{aligned}$$

to give $f(3, 3) = 75 + 15 = 90$. Note also that

$$f(p,\ 1) = 1 \text{ and } f(1,\ q) = \frac{q(q+1)}{2} \text{ for all } p, q \ge 1. \quad (5)$$

Results (1), (2), (4), and (5) show in fact that $f(m - n, n) = S(m, n)$. It follows from (3) that

$$P(k,\ m,\ n) = \frac{S(m,\ n)(k)_n}{k^m}. \qquad (6)$$

We have thus made our connection between the two classroom questions.

By using (6), we can obtain the answer to the problem posed earlier—namely, that of calculating the probability that exactly three distinct numbers occur when a fair six-sided die is rolled five times. We require $P(6, 5, 3)$, which, noting that the aforementioned table tells us that $S(5, 3) = 25$, is equal to

$$\frac{S(5,\ 3)(6)_3}{6^5} = \frac{25\cdot6\cdot5\cdot4}{6^5} = \frac{125}{324}.$$

FINAL THOUGHTS

As are Benjamin and Quinn (2006), I am always keen to introduce students to combinatorial ideas in the classroom. The examples presented here are particularly appealing because of the simplicity of the initial questions that inspired them and the way in

which their generalizations turn out to be linked via a well-known family of numbers. Relatively straightforward ideas in mathematics can lead to material that is altogether more complex and demanding yet nonetheless beautiful. Another instance of this phenomenon appears in Griffiths (2010), who shows how introductory work on the prime factorization of integers may be developed into challenging areas of combinatorics. Such work allows us to show students how genuinely deep, as opposed to superficial, connections occur in mathematics. It may also encourage them to look for links among other questions.

In my classroom work with Stirling numbers of the second kind, one contextual interpretation of $S(m, n)$ is as the number of ways m people can be assigned to n indistinguishable tables such that each table has at least one person sitting at it. Some students are able to establish (1) by arranging and then rearranging themselves among tables in the classroom.

In hindsight, of course, the underlying reason why Stirling numbers of the second kind made an appearance in each encounter was that both questions were connected to the enumeration of partitions. However, this fact is not necessarily obvious without a careful analysis of the situation, thus demonstrating further the wonderfully interconnected nature of mathematics.

The moral of this story is that if we spot partitions lurking in the background of some mathematical scenario, there is a good chance that Stirling numbers of the second kind will make an appearance.

BIBLIOGRAPHY

Benjamin, Arthur, and Jennifer Quinn. 2006. "Revisiting Fibonacci and Related Sequences." *Mathematics Teacher* 99 (5): 357–61.

Burton, David. 1998. *Elementary Number Theory.* 6th ed. Singapore: McGraw-Hill.

Cameron, Peter. 1994. *Combinatorics: Topics, Techniques, Algorithms.* Cambridge: Cambridge University Press.

Griffiths, Martin. 2010. "Thematic Mathematics: The Combinatorics of Prime Factorizations." *Teaching Mathematics and Its Applications* 29 (March): 25–40.

Grimmett, Geoffrey, and David Stirzaker. 2001. *Probability and Random Processes.* 3rd ed. Oxford: Oxford University Press.

Sloane, Neil. 2012. The On-line Encyclopaedia of Integer Sequences. https://oeis.org/.

"Stirling Numbers of the Second Kind." 2012. Wikipedia, The Free Encyclopedia. http://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind.

MARTIN GRIFFITHS, martin.griffiths@strath.ac.uk, who has taught both high school and college mathematics, is a lecturer at the University of Ustaoi. He is particularly interested in how we teach exceptionally able secondary school students and undergraduates.

Statement of Ownership, Management, and Circulation

Statement of ownership, management, and circulation (Required by 39 U.S.C. 3685). 1. Publication title: *Mathematics Teacher*. 2. Publication number: 334-020. 3. Filing date: August 31, 2012. 4. Issue frequency: Monthly: August–December/January; February–May. 5. Number of issues published annually: 9. 6. Annual subscription price: $34. 7. Complete mailing address of known office of publication: National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Fairfax County, VA 20191-1502. Contact person: Pamela Ann Grainger, (703) 620-9840, ext. 2167. 8. Complete mailing address of headquarters or general business office of publisher: same as #7. 9. Full names and complete mailing addresses of publisher, editor, and managing editor. Publisher: National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Fairfax County, VA 20191-1502. Editor: Pamela Ann Grainger, 1906 Association Drive, Reston, VA 20191-1502. Managing editor: none. 10. Owner: National Council of Teachers of Mathematics (nonprofit organization), 501(c)3, 1906 Association Drive, Reston, Fairfax County, VA 20191-1502. 11. Known bondholders, mortgagees, and other security holders owning or holding 1 percent or more of total amount of bonds, mortgages, or other securities: none. 12. Tax status. The purpose, function, and nonprofit status of this organization and the exempt status for federal income tax purposes has not changed during preceding 12 months. 13. Publication title: *Mathematics Teacher*. 14. Issue date for circulation data below: July 24, 2012. 15. Extent and nature of circulation. Average no. copies each issue during preceding 12 months. A. Total number of copies: 26,906. B. Paid circulation. (1) Mailed outside-county paid subscriptions stated on PS form 3541: 20,506; (2) mailed in-county paid subscriptions stated on PS form 3541: none; (3) paid distribution outside the mails including sales through dealers and carriers, street vendors, counter sales, and other paid distribution outside USPS: 844; (4) paid distribution by other classes of mail through the USPS: none. C. Total paid distribution: 21,350. D. Free or nominal rate distribution: (1) free or nominal rate outside-county copies included on PS form 3541: none; (2) free or nominal rate in-county copies included on PS form 3541: none; (3) free or nominal rate copies mailed at other classes through the USPS: 415; (4) free or nominal rate distribution outside the mail: 4553. E. Total free or nominal rate distribution: 4968. F. Total distribution: 26,318. G. Copies not distributed: 588. H. Total: 26,906. I. Percent paid: 81%. 15. Extent and nature of circulation. No. copies of single issue published nearest to filing date. A. Total number of copies: 20,990. B. Paid circulation. (1) Mailed outside-county paid subscriptions stated on PS form 3541: 19,217; (2) mailed in-county paid subscriptions stated on PS form 3541: none; (3) paid distribution outside the mails including sales through dealers and carriers, street vendors, counter sales, and other paid distribution outside USPS: 803; (4) paid distribution by other classes of mail through the USPS: none. C. Total paid distribution: 20,020. D. Free or nominal rate distribution: (1) free or nominal rate outside-county copies included on PS form 3541: none; (2) free or nominal rate in-county copies included on PS form 3541: none; (3) free or nominal rate copies mailed at other classes through the USPS: 415; (4) free or nominal rate distribution outside the mail: none. E. Total free or nominal rate distribution: 415. F. Total distribution: 20,435. G. Copies not distributed: 555. H. Total: 20,990. I. Percent paid: 98%. 16. Publication of statement of ownership will be printed in the November 2012 issue of this publication. 17. Signature and title of editor, publisher, business manager, or owner: Pamela Ann Grainger, senior copy editor, August 31, 2012. I certify that all information furnished on this form is true and complete. I understand that anyone who furnishes false or misleading information on this form or who omits material or information requested on the form may be subject to criminal sanctions (including fines and imprisonment) and/or civil sanctions (including civil penalties).

FOR YOUR information

PUBLICATIONS

Chaos: The Science of Predictable Random Motion, Richard Kautz, 2011. 380 pp., $44.95 paper. ISBN 978-0-19-959458-0. Oxford University Press; www.oup.com.

Chaos theory is a scientific phenomenon in which small changes in input can cause large changes in output. After an introduction, the author looks at chaos theory from historical, scientific, and real-world mathematics perspectives. Kautz gives several familiar examples and shows how they relate to chaos theory: Beethoven's fifth symphony; the butterfly effect; and Tilt-a-Whirl, a popular amusement park ride. In addition, he looks at advanced science concepts and mathematics formulas.

The author's research on science history and the professional life of the scientists included here has been extensive. The conclusion challenges future scientists to continue to search applications of chaos theory. The software that comes with the book also has great graphics and animations.

The book does not include any explanation or activities for the mathematics included. As a result, it is not so useful for high school students, who usually need more guidance on mathematics.

This book would make a great supplemental text for high school AP physics courses and for college physics and engineering courses.

—Sarah Chou
University of Central Florida
Orlando, FL

Crafting by Concepts: Fiber Arts and Mathematics, Sarah-Marie Belcastro and Carolyn Yackel, eds., 2011. 264 pp., $39.95 cloth. ISBN 978-1-56881-435-3; Catalog no. K13062. A K Peters; www.akpeters.com.

The beauty of the visual patterns of mathematics is highlighted in this engaging book. The mathematics is carefully laid out, but following every detail of the mathematics is not necessary to create the patterns of the crafts projects presented. Readers can find projects for many different skills—knitting, tatting, quilting, needlepoint, cross-stitch, crochet, and temari. Included for each project are complete instructions for experienced crafters as well as adaptations for mathematics teachers to use with students from kindergarten to graduate school.

Imagine a cross-stitched representation of the elements, subgroups, and cosets of the symmetry group D_4! This is one of the first examples of the stitchable patterns presented; others involve reflections, rotations, translations, and glide reflections. A later example is wallpaper patterns. Not all wallpaper patterns can be demonstrated in cross-stitch because cross-stitch is based on a square grid, and some wallpaper patterns use 60-degree angles. But all wallpaper patterns that can be stitched are shown in photographs.

Excellent photographs throughout amply illustrate the use of mathematics to determine the patterns and also illuminate some hard-to-describe stitching methods—for example, producing various tiling patterns through quilting.

I highly recommend this book to teachers looking for beautiful projects for their students or to anyone who loves or is curious about circles, spirals, helixes, cross-caps, fractals, diaper patterns, symmetry groups, squaring the rectangle, and tiling spheres or the plane.

—Marcia Weller Weinhold
Purdue University Calumet
Hammond, IN

Mathematics Education for a New Era: Video Games as a Medium for Learning, Keith Devlin, 2011. 218 pp., $29.95 paper. ISBN 978-1-56881-431-5. A K Peters; www.akpeters.com.

Imagine a teaching approach that included the following:

- Conceptual learning mixed with practice
- Appropriate rewards for success with mild penalties for failure (that can easily be atoned for)
- Students working at their own pace with encouragement to work collaboratively

Imagine also that millions of students eagerly work through these activities on their own time.

This is how Keith Devlin perceives games such as World of Warcraft (WOW). Unfortunately, what students learn in these games is not well aligned with the curriculum. Devlin's goal is to change that. Why not develop games that teach mathematics concepts using the same successful techniques as WOW?

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

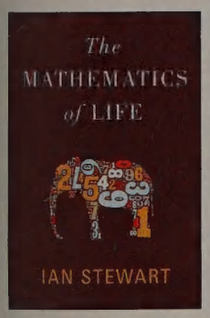
Devlin's argument, framed in the context of his philosophy of teaching mathematics, is extremely thought provoking and well worth reading. I do not agree with everything he proposes, particularly concerning the relationship between practice and understanding, but his ideas have helped me grow in my own philosophy of teaching.

The dreams expressed here can help counteract the nightmare of turning schools into testing factories. The book has at its core an idea whose time has come—to implement games that include mathematically significant content beyond the mostly drill games that teachers currently use.

I highly recommend this book to anyone interested in reflecting on why and how we teach mathematics as well as how we could bring some of the energy that students expend in video games to the mathematics classroom.

—Larry Feldman
Indiana University of Pennsylvania
Indiana, PA

The Mathematics of Life, Ian Stewart, 2011. 368 pp., $27.99 cloth. ISBN 978-0-465-02238-0. Basic Books; www.basicbooks.com.



This book presents a detailed yet accessible look at biologists' and mathematicians' collaborative efforts to explore problems in the life sciences. Writing to a general audience, Stewart succeeds in not only describing mathematical patterns found in biological structures, from Fibonacci-based patterns in plants to the geometry of viruses, but also accounting for *why* these patterns occur.

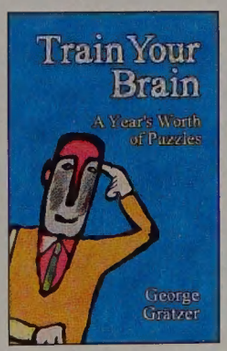
The first seven chapters summarize the history of biology, organized around five "revolutions" that have significantly expanded our study and understanding of it: the microscope, classification schemes of living creatures, evolution, genetics, and DNA structure. This organization offers a refreshing (and, for me at least, a much needed) review of biology and sets the stage for Stewart's sixth revolution: mathematics.

The remaining chapters describe mathematical work on specific problems in biology, including why some animals have stripes and others spots, how the structure of nerve cell groups dictates the way animals move, how DNA molecules replicate, and what extraterrestrial life might look like. Stewart also introduces readers to the basics of relevant branches of mathematics, such as probability, game theory, topology, Euclidean geometry, and chaos.

The Mathematics of Life offers a fascinating glimpse of mathematics at biology's cutting edge. The wide scope of topics—both mathematical and biological—keeps the narrative consistently interesting. Well researched and well written, this book is a great summer reading option for mathematics teachers.

—Kyle T. Schultz
James Madison University
Harrisonburg, VA

Train Your Brain: A Year's Worth of Puzzles, George Gratzer, 2011. 253 pp., $29.95 paper. ISBN 978-1-56881-710-1; Cat. no. K00594. A K Peters and CRC Press; www.akpeters.com.



This book is intended for "high-school students who want to learn deductive-reasoning skills, for adults who enjoy challenging their brains, and for older people who want to stay mentally fit." This collection of puzzles has been constructed with the intention that the reader will take each puzzle in turn and let it simmer in his or her mind. With a total of 140 puzzles, the book allows readers two to three days per situation. If stumped, readers can turn to the back of the book for good explanations of methods for obtaining the solution.

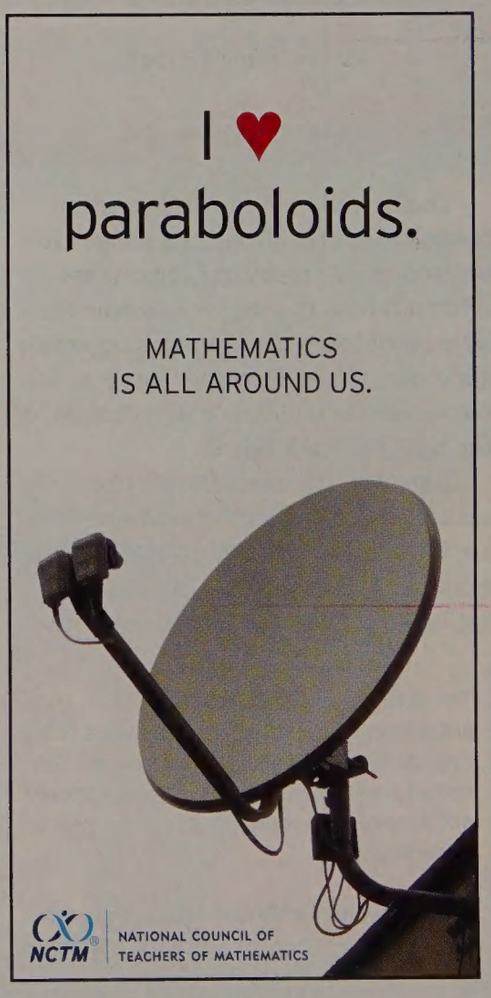
Another feature of this collection is alternate versions of puzzles. Many twist one of the puzzle's parameters, creating a familiar but novel and interesting problem. For each of the fifty-two weeks, I found at least one new puzzle that I had not seen before. Hints are provided, but I tended not to look at them; when I did, they were of little help.

This puzzle collection is a great assortment of mathematical-logical thought cultivators. It presents novel puzzles interspersed with a handful of classics. These challenging brainteasers will set students on a path of deductive reasoning.

—Michael Davis
Sebastian River High School
Sebastian, FL

GIVE US YOUR SUGGESTIONS

If you know of a new book or product that you would like to see reviewed in *MT*, let us know. Send information about the item, including its title and the publisher's name, to us at **mtreviews@nctm.org.** We will be sure to consider the item for FYI.



MY FAVORITE lesson

William J. Wolbert

Sliding through Logarithms

One of my favorite lessons involves a nostalgic look at the past, in the years BC (Before Calculators). The properties of logarithms are readily reinforced using an ingenious tool from the past—the slide rule. Developed in the seventeenth century by William Oughtred and others, who built on the emerging work on logarithms by John Napier, the slide rule was used to multiply, divide, and find square roots and cube roots, among other arithmetic operations. Before the advent of the pocket calculator, it was the most commonly used calculation tool in science and engineering.

This lesson involves the properties given by the following identities:

$$\log(ab) = \log a + \log b$$

$$\log\left(\frac{a}{b}\right) = \log a - \log b$$

$$\log\left(a^b\right) = b\log a$$

The base chosen for the logarithm is irrelevant. The properties above are written using "common" logarithms. After a few examples, we examine the seven-foot-long classroom demonstrator slide rule, focusing on the C scale at the bottom of the middle section, or slide, of the slide rule (see **fig. 1**).

Students will generally notice that the scale isn't "consistent": The 2 and the 3 are farther apart than the 6 and the 7, for example. I have students find log 2 and log 3 on their calculators and compare

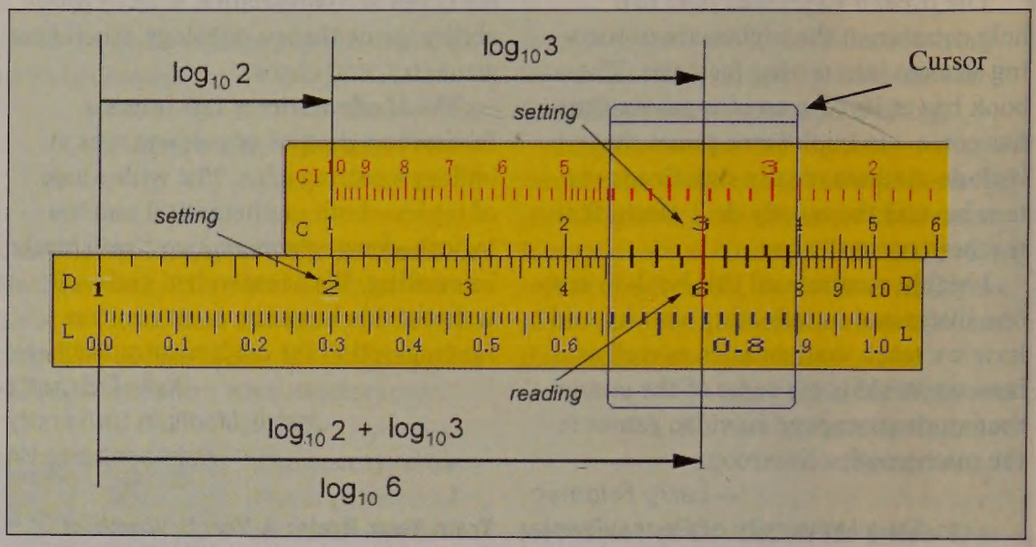


Fig. 1 Multiplication on the slide rule uses the C and D scales.

their difference with the difference of log 6 and log 7. Students will find that log 3.16 is very close to 0.5; hence, 3.16 is centered on the C scale. The scale represents the logarithms of the printed numbers. The 1 at the left is log 1 = 0, and the 10 at the left represents log 10 = 1.

Now, if we duplicate this scale, called scale D, on another slide and align it appropriately along the C scale, we will be adding lengths of these two scales—in effect, adding logarithms and multiplying the corresponding numbers.

Students can see that to multiply two numbers—say, 2 and 3—we manipulate the slides so that log 2 is "extended" by log 3: We align the 1 on the C scale with the 2 on the D scale and move the cursor to the 3 on the C scale. The answer, 6, is seen under the hairline on the D scale. In essence, we added a length of log 2 to a length of log 3 to get the length of log 6. This calculation is shown on the portion of the slide rule containing the C and D scales.

We then investigate division of two numbers by subtracting their logarithms using a process similar to multiplication, thereby reinforcing $\log(a/b) = \log a - \log b$. To demonstrate the property that $\log(a^b) = b \cdot \log a$, we examine the A and K scales. To square 2, for example, we need only double its logarithm; to cube 2, we need only triple its logarithm. Using the D scale in conjunction with the A and K scales accomplishes these calculations, showing that $2^2 = 2 \cdot \log 2$ and $2^3 = 3 \cdot \log 2$. To find a square root of a number, we find that number on the A scale using the cursor and read its square root on the D scale under the hairline. Cube roots are found using the K and D scales in the same way.

Students are enthralled by this step back into the past, especially when I explain that the slide rule was my sole calculating device when I was in high school. This lesson also provides a hands-on activity to help reinforce the properties of logarithms.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

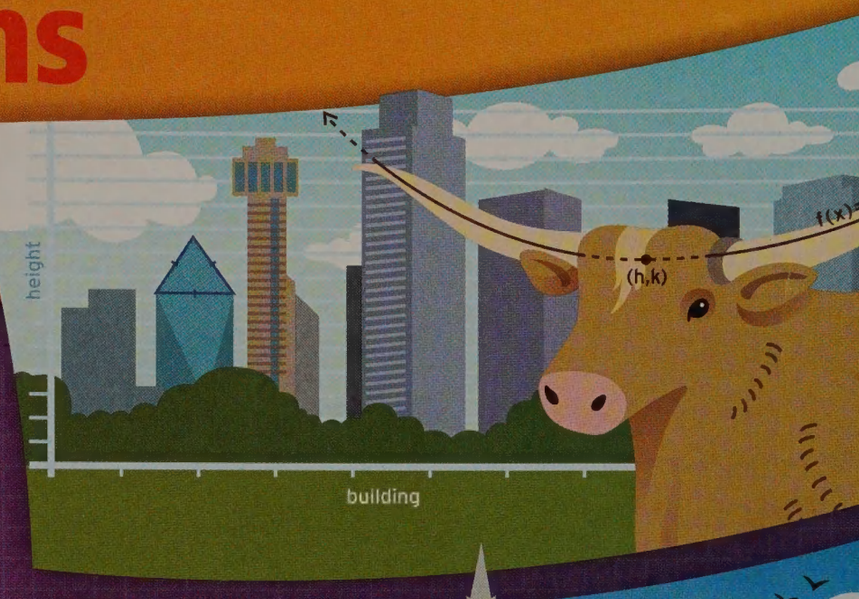
Edited by ***Jennifer Wexler,*** *wexlerj@newtrier.k12.il.us, New Trier High School, Winnetka, IL*

WILLIAM J. WOLBERT, pawwjw@hughes.net, retired after thirty-seven years of teaching high school mathematics at North Clarion School in Tionesta, Pennsylvania, and at Clarion University in Clarion.

NCTM 2012

Regional Conferences & Expositions

DALLAS, TX | OCTOBER 10–12
HARTFORD, CT | OCTOBER 24–26
CHICAGO, IL | NOVEMBER 28–30

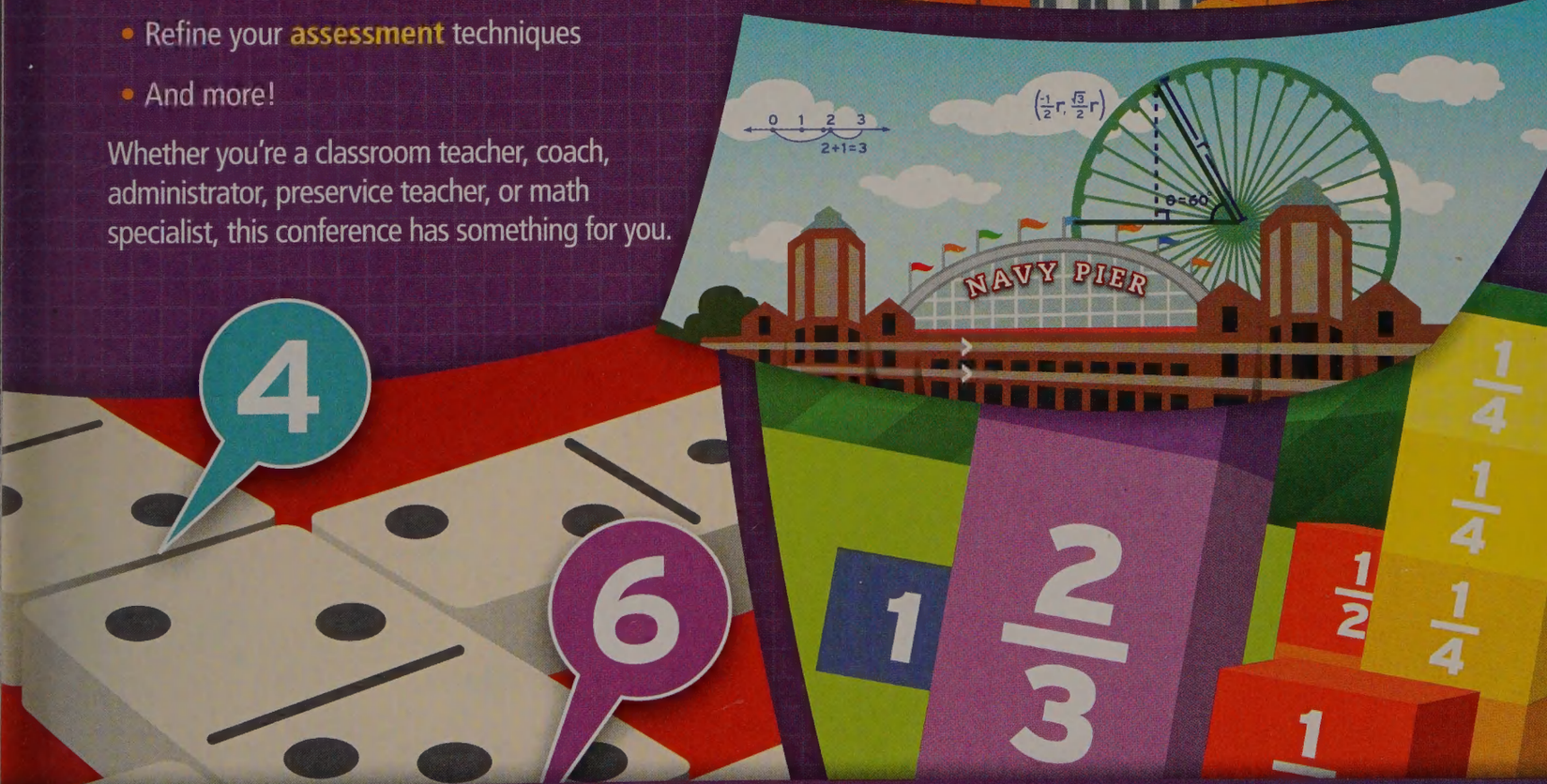


Join the Best & Brightest Math Educators

NCTM's Regional Conferences have the professional development opportunities you need to help your students succeed. Join the best and the brightest and immerse yourself in the latest topics in math education. By attending you and your colleagues will:

- Learn more about and test the latest **technology**
- Discover new and effective **intervention** methods
- Learn practices central to teaching the **Common Core State Standards**
- Refine your **assessment** techniques
- And more!

Whether you're a classroom teacher, coach, administrator, preservice teacher, or math specialist, this conference has something for you.



NCTM
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
(800) 235-7566 | WWW.NCTM.ORG

Visit **www.nctm.org/regionals** to learn more and register.

New Books and Products from NCTM

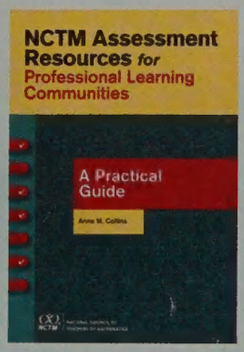
NEW FALL 2012 BOOKS

NEW

Curriculum Issues in an Era of Common Cores State Standards for Mathematics

By Christian Hirsch

Stock # 14319
List Price: $37.95
Member Price: $30.36



NEW

NCTM Assessment Resources for the Professional Learning Community: A Practical Guide

By Anne Collins

Stock # 14093
List Price: $20.95
Member Price: $16.76

Just Published by Frances Stern

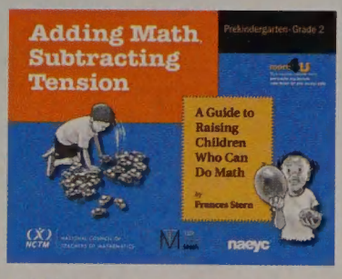
Adding Math, Subtracting Tension: Grades 3-5

By Frances Stern

Stock # 13799

NCTM Members always SAVE 20% off all books and products!

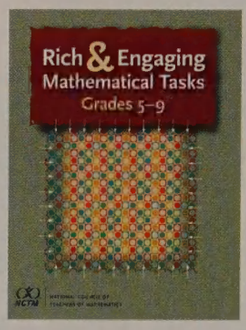
Also by Frances Stern



Adding Math, Subtracting Tension: A Guide to Raising Children Who Can Do Math—Prekindergarten–Grade 2

Stock # 13786 | List Price: $26.95 | **Member Price: $21.56**

MORE NEW TITLES

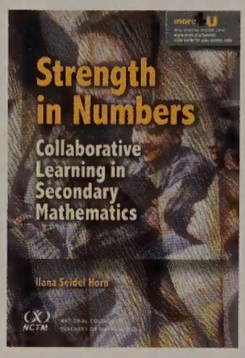


NEW

Rich and Engaging Mathematical Tasks: Grades 5–9

Glenda Lappan

Stock # 13516
List Price: $36.95
Member Price: $29.56

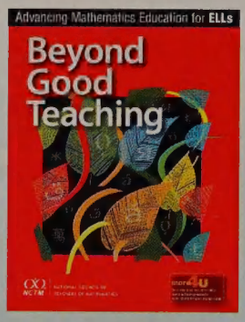


NEW

Strength in Numbers: Collaborative Learning in Secondary Mathematics

Ilana Horn

Stock # 13791
List Price: $29.95
Member Price: $23.96

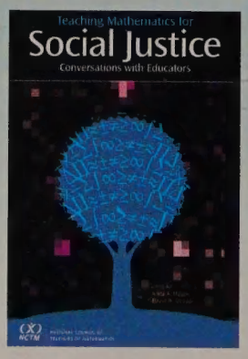


NEW

Beyond Good Teaching: Advancing Mathematics Education for ELLs

By Nora Ramírez and Sylvia Celedón-Pattichis

Stock # 14118
List Price: $35.95
Member Price: $28.76



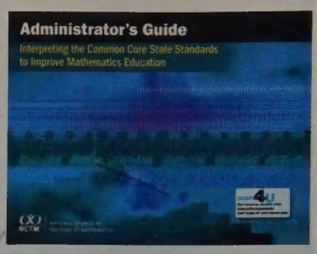
NEW

Teaching Mathematics for Social Justice: Conversations with Educators

Edited by David Stinson and Anita Wager

Stock # 13955
List Price: $35.95
Member Price: $29.56

MATHEMATICS BOOKS BY AND FOR MATHEMATICS EDUCATORS



Administrator's Guide: How to Interpret the Common Core State Standards to Improve Mathematics Education

By Matt Larson

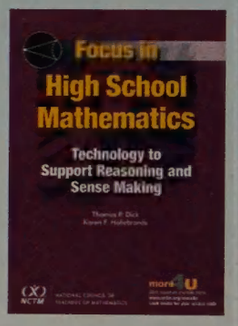
Stock # 14288
List Price: $23.95 | **Member Price: $19.16**

Reasoning and Sense-Making Problems and Activities for Grades 5–8

Edited by Elizabeth D. Phillips and Judith S. Zawojewski

Stock # 13497
List Price: $37.95
Member Price: $30.36

Last Title in Series!



Focus in High School Mathematics: Technology to Support Reasoning and Sense Making

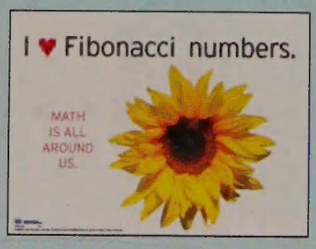
By Thomas P. Dick and Karen Hollebrands

Stock # 14287
List Price: $32.95
Member Price: $26.36

Books with More4U logo have additional resources online. Look inside book for access code.

www.nctm.org/more4u

I HEART POSTERS— *Beautiful and Smart*



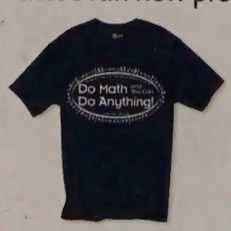
Available as a set of ten or sold individually.

List Price: $10.95 each

Member Price: $8.76 each

NEW PRODUCTS!

NCTM Members always SAVE 20% off all books and products. Show the world your love of math with these fun new products!



Navy DO MATH T-shirt

In five sizes, S—XXL

List Price: $16.95
Member Price: $13.95

DO MATH Tote Bag

Stock # 14394
List Price: $12.95
Member Price: $10.36

DO MATH Lunch Tote

Stock # 14393
List Price: $6.95
Member Price: $5.56

*Visit **www.nctm.org/catalog** for tables of content and sample pages.*

NCTM® NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

For more information or to place an order, please call **(800) 235-7566** or visit **www.nctm.org/catalog**.